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TIME ACCURATE FAST THREE-STEP WAVELET-GALERKIN METHOD FOR PARTIAL DIFFERENTIAL EQUATIONS

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We introduce the concept of three-step wavelet-Galerkin method based on the Taylor series expansion in time. Unlike the Taylor–Galerkin methods, the present scheme does not contain any new higher-order derivatives which makes it suitable for solving nonlinear problems. Numerical schemes taking advantage of the wavelet bases capabilities to compress the operators and sparse representation of functions which are smooth, except for localized regions, up to any given accuracy are presented. Here numerical experiments deal with advection equation with the spiky solution in one dimension, two dimensions and nonlinear equation with a shock in solution in two dimensions. Numerical results indicate the versatility and effectiveness of the proposed scheme.

Keywords: Three-step method; wavelets; parabolic equation; hyperbolic equation.

AMS Subject Classification: 65M70, 35-xx

1. Introduction

Wavelets method is a new numerical tool for solving partial differential equations (PDEs). Wavelet analysis assumed significance due to successful applications in signal and image processing during the eighties. Compactly supported wavelets which are differentiable were introduced by Daubechies in her celebrated paper,[1] which has had applications in a number of areas. Since these functions combine orthogonality with localization and scaling properties, it has been a natural idea to attempt to use these functions for the numerical approximation to solutions to partial differential equations (PDEs). There have been a number of papers in this direction including studies of Burgers' equation[2,3] as well as two-dimensional flow.[4–6] Glowinski *et al.*[7] considered wavelet-based variational methods to solve one-dimension linear and nonlinear ordinary differential equations. Where multiresolution analysis and their associated scale function bases are used as alternative bases in Galerkin methods.

Thus, such methods have convergence properties similar to the ones of spectral methods, and simultaneously, partial derivative operators discretize similarly as the finite-difference methods.

In the literature, many tentatives have been performed, often based on Galerkin or Petrov–Galerkin methods which use the compression properties of wavelet bases, and contain specific wavelet methods for PDEs. Some of them take advantage of the wavelet compression of the solution,[2] others use instead the wavelet compression of the operator.[8] The aim of this paper is to introduce fast three-step wavelet-Galerkin method which has the benefit of both these properties.

In the conventional numerical approach to transient problems, the accuracy gained in using the high-order spatial discretization is partially lost due to the use of low-order time discretization schemes. Here, usually spatial approximation precedes the temporal discretization. On the contrary, the reversed order of discretization can lead to better time accurate schemes with improved stability properties. The fundamental idea behind the Taylor–Galerkin approach[9] is the substitution of space derivatives for the time derivatives in the Taylor series, as used in derivation of the Lax–Wendroff method,[10] the only modification being that the procedure is carried out to third order. However, its applications are mainly for hyperbolic problems and some convection diffusion equations, because too many terms are introduced in the third-order time derivative term, especially for nonlinear multidimensional equations, and treatment of the boundary integrations arising from high-order time derivative terms are too complicated.

A three-step finite element method based on a Taylor series expansion in time is proposed in Ref. 11. This scheme involves neither complicated expression nor higher-order derivatives like the Taylor–Galerkin method[9] but the advantage has not been taken of sparsity of matrices which are in evolutionary problems. Time accurate solution of Korteweg–de Vries equation using the wavelet-Galerkin method is also developed in Ref. 12 and wavelet multilayer Taylor–Galerkin schemes for hyperbolic and parabolic problems are introduced in Ref. 13. Our three-step wavelet-Galerkin method is based on fast algorithms like matrix vector product in wavelet bases and wavelet compression property of smooth data.

Solutions to PDEs often behave differently in different areas. In fluid dynamics, we have shocks, boundary layers and turbulence. In acoustics, an example of PDEs is a low-frequency wave, with a localized high-frequency burst. For these examples, the solution can be smooth in most of the solution domain, with small areas where the solution changes quickly. Where the discretization in space (correlatively in time) ought to handle a huge number of degrees-of-freedom. The aim of the present paper is to examine the feasibility of applying fast three-step wavelet-Galerkin method to such types of PDEs which require the advantage of wavelet compression. A combination of such a time-marching scheme with wavelet approximation in space can lead to simple higher-order space and time accurate numerical methods.

2. Wavelet Preliminaries

Compactly supported wavelets have several properties that are quite useful for representing solutions of PDEs. The orthogonality, compact support and exact representation of polynomials of a fixed degree allow the efficient and stable calculation of regions with strong gradients or oscillations. Daubechies defines the class of compactly supported wavelets.[1] Briefly, let ϕ be a solution of the scaling relation

$$\phi(x) = \sum_k a_k \phi(2x - k).$$

The a_k are a collection of coefficients that categorize the specific wavelet basis. The expression ϕ is called the scaling function. The associated wavelet function ψ is defined by the equation

$$\psi(x) = \sum_k (-1)^k a_{1-k} \phi(2x - k).$$

The normalization $\int \phi dx = 1$ of the scaling function leads to the condition $\sum_k a_k = 2$. The translates of ϕ are required to be orthonormal, i.e. $\int \phi(x-k)\phi(x-m)dx = \delta_{k,m}$. The scaling relation implies the condition $\sum_{k=0}^{N-1} a_k a_{k-2m} = \delta_{0,m}$ where N is the order of wavelet. For coefficients verifying the above two conditions, the functions consisting of translates and dilations of the wavelet function, $\psi(2^j x - k)$, form a complete, orthogonal basis for square integrable functions on the real line, $L^2(R)$.

If only a finite number of the a_k are nonzero, then ϕ will have compact support. Since

$$\int \phi(x)\psi(x-m)dx = \sum_k (-1)^k a_{1-k} a_{k-2m} = 0,$$

the translates of the scaling function and wavelet define orthogonal subspaces

$$V_j = \{2^{j/2}\phi(2^j x - k); m = \ldots, -1, 0, 1, \ldots\},$$
$$W_j = \{2^{j/2}\psi(2^j x - k); m = \ldots, -1, 0, 1, \ldots\}.$$

The relation

$$V_{j+1} = V_j + W_j$$

implies the Mallat transform[1]

$$V_0 \subset V_1 \subset \cdots \subset V_{j+1},$$
$$V_{j+1} = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_j.$$

Smooth scaling functions arise as a consequence of the degree of approximation of the translates. The result that the polynomials $1, x, \ldots, x^{p-1}$ be expressed as linear combinations of the translates of $\phi(x-k)$ is implied by the conditions

$$\sum_k (-1)^{-k} k^m a_k = 0$$

for $m = 0, 1, \ldots, p-1$. The following are equivalent results.

(i) $\{1, x, \ldots, x^{p-1}\}$ are linear combinations of $\phi(x-k)$;
(ii) $\left\| f - \sum c_k^j \phi(2^j x - k) \right\| \leq C 2^{-jp} \|f^p\|$, where $c_k^j = \int f(x)\phi(2^j x - k)dx$;
(iii) $\int x^m \psi(x)dx = 0$ for $m = 0, 1, \ldots, p-1$;
(iv) $\int f(x)\psi(2^j x)dx \leq c 2^{-jp}$.

For the Daubechies scaling/wavelet function, DN have $p = N/2$. In Fig. 1, we see an example of a compactly supported scaling function and its associated fundamental wavelet function. By rescaling and translation, we obtain a complete orthonormal system for $L^2(R)$ which has a sufficient smoothness to also be a basis for $H^1(R)$. This wavelet system then yields a basis for solution methods for second-order elliptic boundary problems on intervals on the real line. The illustrated example has fundamental support $[0, 5]$. For arbitrarily large even N, there is the Daubechies example of a fundamental scaling function defining a wavelet family with support in the interval $[0, N-1]$.[1] As pointed out by Meyer (1990), the complete toll box built in $L^2(R)$ can be used in the periodic case $L^2([0,1])$

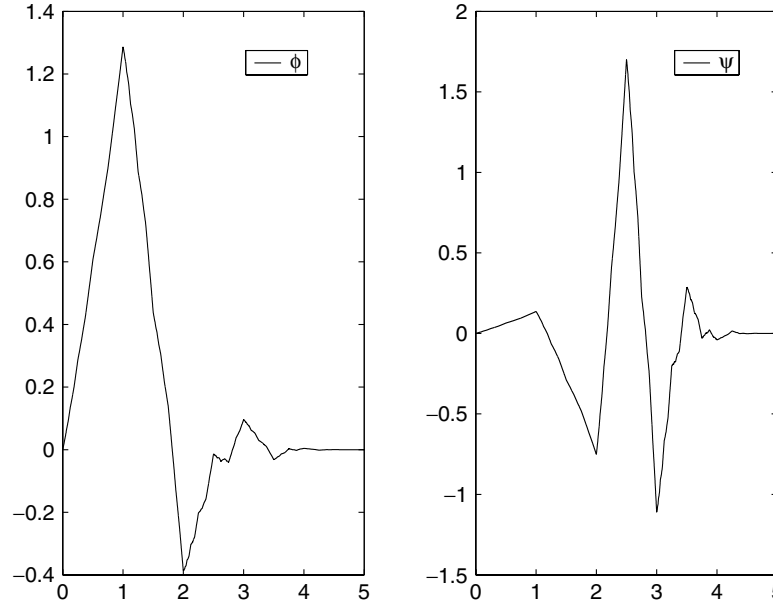

Fig. 1. Daubechies scaling and wavelet functions for $N = 6$ with support on $[0, 5]$.

by introducing a standard periodization technique. This technique consists at each scale in folding, around the integer values, the wavelet $\psi_{j,k}$ and the scaling functions $\phi_{j,k}$ centered in $[0,1]$. It writes $\tilde{\phi}_{j,l}(x) = \sum_{n=-\infty}^{\infty} \phi_{j,l}(x+n)$ and $\tilde{\psi}_{j,l}(x) = \sum_{n=-\infty}^{\infty} \psi_{j,l}(x+n)$ and generates V_{Pj} and W_{Pj}. A function $f \in V_{Pj}$ in pure periodic scaling function expansion $f(x) = \sum_{k=0}^{2^j-1} c_k^j \tilde{\phi}_{j,k}(x)$ and the periodic wavelet expansion $f(x) = \sum_{k=0}^{2^{J_0}-1} c_k^{J_0} \tilde{\phi}_{J_0,k}(x) + \sum_{j=J_0}^{J-1} \sum_{k=0}^{2^j-1} d_k^j \tilde{\psi}_{j,k}(x)$; where J_0 satisfies $0 \le J_0 \le J$ and the decay of the wavelet coefficient is given by the following theorem[14]:

Theorem 2.1. *Let $P = D/2$ be the number of vanishing moments for a wavelet $\psi_{j,k}$ and let $f \in C^P(R)$. Then the wavelet coefficients decay as $|d_{j,k}| \le C_P 2^{-j(P+\frac{1}{2})} \max_{\xi \in I_{j,k}} |f^{(P)}(\xi)|$.*

3. Three-Step Wavelet-Galerkin Method

Before introducing three-step method, it is necessary to have a brief statement of the two-step Lax–Wendroff method. Performing a Taylor series expansion in time, we have

$$u(t+\delta t) = u(t) + \delta t \frac{\partial u(t)}{\partial t} + \frac{\delta t^2}{2} \frac{\partial^2 u(t)}{\partial t^2} + \frac{\delta t^3}{6} \frac{\partial^3 u(t)}{\partial t^3} + O(\delta t^4). \tag{3.1}$$

By approximating Eq. (3.1) to second-order accuracy, the formulation of two-step method can be derived as

$$\begin{aligned} u\left(t + \frac{\delta t}{2}\right) &= u(t) + \frac{\delta t}{2} \frac{\partial u(t)}{\partial t}, \\ u(t+\delta t) &= u(t) + \delta t \frac{\partial u(t+\delta t/2)}{\partial t}. \end{aligned} \tag{3.2}$$

Now, we introduce the three-step method.

3.1. *Principle of T-WGM for advection-diffusion equation*

Consider the linear advection-diffusion equation

$$\partial_t u = -a \partial_x u + \nu \partial_x^2 u, \tag{3.3}$$

where a and $\nu > 0$ are positive constant coefficients.

Let us leave the spatial variable x continuous and discretize (3.3) in time. To obtain an improved order of accuracy in δt, we shall apply a three-step method based on the Taylor series expansion (3.1). By approximating Eq. (3.1) up to third-order accuracy, the formulations of the three-step method can be written as

$$\begin{aligned} u\left(t + \frac{\delta t}{3}\right) &= u(t) + \frac{\delta t}{3} \frac{\partial u(t)}{\partial t}, \\ u\left(t + \frac{\delta t}{2}\right) &= u(t) + \frac{\delta t}{2} \frac{\partial u(t+\delta t/3)}{\partial t}, \\ u(t+\delta t) &= u(t) + \delta t \frac{\partial u(t+\delta t/2)}{\partial t}. \end{aligned} \tag{3.4}$$

In Eq. (3.4), by replacing the time derivative with spatial derivatives, the associated three-step wavelet-Galerkin equations give the T-WGM scheme of

$$\begin{aligned} u^{n+\frac{1}{3}} &= u^n + D_1 u^n, \\ u^{n+\frac{1}{2}} &= u^n + D_2 u^{n+\frac{1}{3}}, \\ u^{n+1} &= u^n + D_3 u^{n+\frac{1}{2}}, \end{aligned} \tag{3.5}$$

where the operator D_1 will look like $D_1 = \frac{\delta t}{3}\big(-a\partial_x + \nu\partial_x^2\big)$. Now, the wavelet-Galerkin discretization turns the problem into a finite-dimensional space.

$$\begin{aligned} d_u^{n+\frac{1}{3}} &= d_u^n + \mathcal{D}_1 d_u^n, \\ d_u^{n+\frac{1}{2}} &= d_u^n + \mathcal{D}_2 d_u^{n+\frac{1}{3}}, \\ d_u^{n+1} &= d_u^n + \mathcal{D}_3 d_u^{n+\frac{1}{2}}. \end{aligned} \tag{3.6}$$

In this finite-dimensional space, u^n is to be replaced by the vector d_u^n along a wavelet finite basis, and D are replaced by, respectively, $\mathcal{D}$ (finite) matrices. Now, to solve Eq. (3.6) in the wavelet basis, we will compute $\mathcal{D}$ once and store in compressed form. We can now give a computational procedure for computing (3.6) using wavelet compression.

Algorithm 1

(i) $trunc(\mathcal{D}, \epsilon_M)$ --→ $(\mathcal{D})^{\epsilon_M}$
(ii) compute initial guess in wavelet basis --→ d_u^0
(iii) $trunc\big(d_u^0, \epsilon_V\big)$ --→ $\big(d_u^0\big)^{\epsilon_V}$
(iv) for $n = 0, 1, \ldots, n1 - 1$
(v) $\mathcal{D}^{\epsilon_M}\big(d_u^n\big)^{\epsilon_V}$ --→ d_u^{n+1}
(vi) $trunc\big(d_u^{n+1}, \epsilon_V\big)$ --→ $\big(d_u^{n+1}\big)^{\epsilon_V}$

where $trunc(d_u, \epsilon_V) = \big\{d_k^j, |d_k^j| > \epsilon_V\big\}$ and $trunc(\mathcal{D}, \epsilon_M) = \{[\mathcal{D}_{m,n}], [\mathcal{D}_{m,n}] > \epsilon_M\}$.

A further property of the wavelet representation of operators is that the successive powers $\mathcal{D}^n$ of the time iteration matrix become sparser and sparser with increasing n. This property is very specific to wavelets, as the opposite occurs with finite difference where $\mathcal{D}^n$ becomes a more and more dense matrix as shown in Fig. 2. It is seen from Fig. 2 that in the wavelet-Galerkin approach, compression in the matrix $\mathcal{D}^n$ is larger than finite difference approach. From this property, we can obtain iterative speed of the three-step wavelet-Galerkin scheme.

Algorithm 2

(i) Initialize $(\mathcal{D}_0)^{\epsilon_M}$ and $\big(d_u^0\big)^{\epsilon_V}$
(ii) for $n = 0, 1, \ldots, n1 - 1$
(iii) $(\mathcal{D}_n)^{\epsilon_M}\big(d_u^n\big)^{\epsilon_V}$ --→ $\big(d_u^{n+1}\big)^{\epsilon_V}$
(iv) $\mathcal{D}_n^2$ --→ $\mathcal{D}_{n+1}$.
(v) endfor

Then the approximate solution of PDE is at $t = 2^n \delta t$ is $d_u^{(2^n)}$.

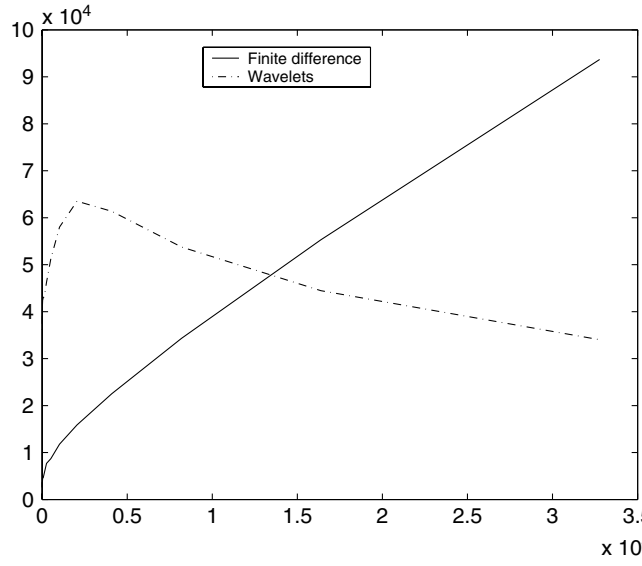

Fig. 2. Number of coefficients in the successive powers of $\mathcal{D}^n$ based on CN times stepping in wavelets and in finite differences, versus $x = 2^n$, $n = 15$, $N = 1024$, $\delta t = 10^{-3}$, $\epsilon_M = 10^{-8}$.

There is no need to change from classical to wavelet coordinates until some time steps. In classical coordinates, the evolution operator changes from very sparse to dense. In the wavelet representation, we may start the squaring in the classical coordinates and change to the wavelet basis at the point where the wavelet representation is sparser. Thus, we have the following algorithm:

Algorithm 3

(i) for $n = 0, 2, \ldots, p$
(ii) $\mathcal{D}u^n \dashrightarrow u^{n+1}$
(iii) endfor
(iv) Initialize $\mathcal{D}^{\epsilon_M}$ and $\left(d_u^p\right)^{\epsilon_V}$
(v) for $n = p+1, p+2, \ldots, p+n1-1$
(vi) $\mathcal{D}^{\epsilon_M}\left(d_u^n\right)^{\epsilon_V} \dashrightarrow \left(d_u^{n+1}\right)$
(i) $trunc\left(d_u^{n+1}, \epsilon_V\right) \dashrightarrow \left(d_u^{n+1}\right)^{\epsilon_V}$
(ii) endfor

It is essential for the success of these algorithms that the computation of the matrix vector product fully exploits the compressed form of both matrix and vector. This can be done using the algorithm in Ref. 14 or fast multiplication based on a general sparse format for both matrix and vector. Our technique is more favorable for parabolic problems in terms of taking the advantage of sparsity of operator. For hyperbolic (1st-order systems of) PDEs, the situation is different. We no longer have a sparse representation of the operator in wavelet space for the hyperbolic

problem. However, a characteristic for hyperbolic problems is the presence of shocks in the solution. A solution might be smooth and nearly constant over the whole domain except at one point where it is discontinuous. Therefore, for hyperbolic problems, wavelets could be an efficient way to represent the solution instead of, the solution operator for parabolic and elliptic problems. In contrast with the Taylor–Galerkin method, the three-step method does not contain any new higher-order spatial derivatives and can thus be applied to solve nonlinear multidimensional flows with easy.

3.1.1. *Theoretical stability of the linearized schemes*

We use the notion of asymptotic stability of a numerical method as it is defined in Ref. 15 for a discrete problem of the form $du/dt = Lu$ where L is assumed to be a diagonal matrix. This is because for many evolution equations, it is necessary to adapt the time steps to the spatial resolution in order to maintain the stability and precision of the numerical scheme. The region of absolute stability of a numerical method is defined for the scalar model problem $du/dt = \lambda u$ to be set of all $\lambda\delta t$ such that $||u^n||$ is bounded as $t \to \infty$. Finally, we say that a numerical method is asymptotically stable for a particular problem if, for small $\delta t > 0$, the product of δt times every eigenvalues of L lies within the region of absolute stability.

3.1.2. *Numerical simulation of advection equation*

Here, we examine the performance of the T-WGM on a linear advection equation on the unit interval[17] with periodic boundary conditions. The initial condition $u_0(x) = \sin(2\pi x) + \exp^{-\alpha(x-1/2)^2}$, which is smooth in most of the domain except near $x = 0.5$, where we have a spike and the thresholded wavelet expansion of the solution $u(x,t)$ is shown in Fig. 3 for $\alpha = 10^4$. This wavelet expansion will have few coefficients, except for in the neighborhood of the spike at $x = 0.5 - t$. We stepped forward to $t = 0.3$, where the solution for $j = 8$ and $j = 9$ is shown in Fig. 4 using wavelet-Galerkin method (WGM). Here, oscillation are produced throughout the domain due to the quick change of solution near spike for $j = 8$ and oscillation are confined near the neighborhood of spike for $j = 9$. In terms of finite difference methods, we want to have many points in areas where the solution has strong variation and few points in area where the solution is smooth. If we use a Galerkin method, this corresponds to the representation of the solution having fewer basis functions in the smooth areas. Note that by thresholding a wavelet representation, we have a way to automatically find a sparse representation of smooth part. Holmstrom and Walden have applied adaptive wavelet methods on such type of PDEs.[18,17] However, by our approach of T-WGM scheme, we are taking the advantage of the time accurate scheme as well as wavelet capabilities of compression to produce fast algorithm based on fast matrix vector product in terms of sparsity. We are getting oscillation-free solution as shown in Fig. 5 without and

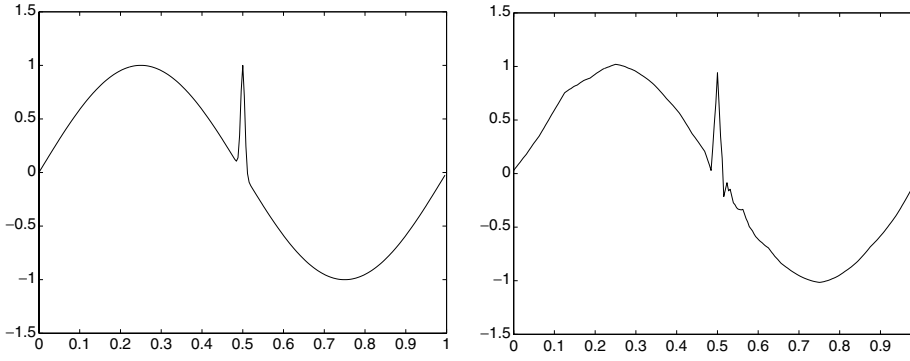

Fig. 3. (a) Initial function $u_0(x)$; (b) Truncated approximate solution by D_6 with the threshold $\epsilon = 0.001$, which leads to 13 retained wavelet coefficients, out of the original 2^8.

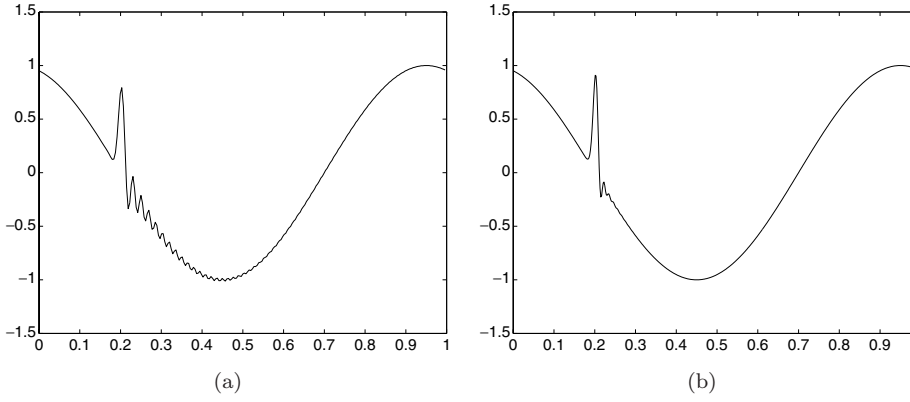

Fig. 4. Solution of advection equation at $t = 0.3$ by WGM.

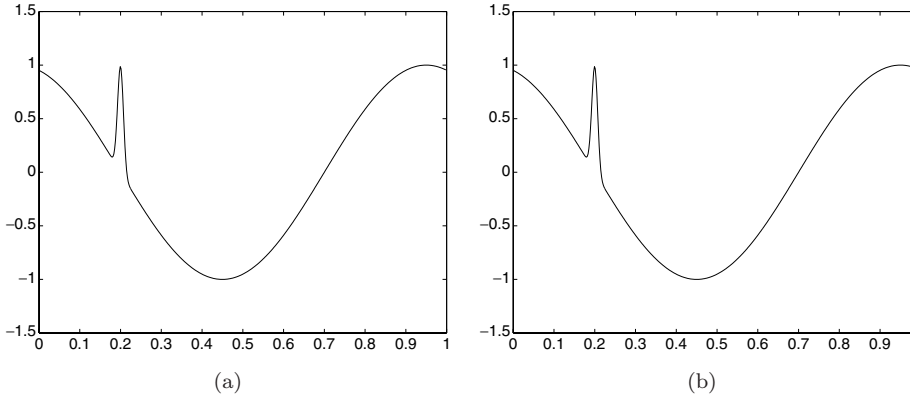

Fig. 5. Solution of advection equation at $t = 0.3$, $j = 9$ by T-WGM.

with truncation, where WGM is unable to produce oscillation-free solution near the spike with the same degrees-of-freedom.

Error in approximated solution for WGM, two-step Lax–Wendroff wavelet Galerkin method (L-WGM) and T-WGM without truncation is shown in Fig. 6; where near the spike error in the T-WGM scheme is small compared to both L-WGM and WGM scheme. From this, we can conclude that near the sharp gradients, we can take the advantage of time accuracy and compression properties of wavelet in the T-WGM scheme. These nice properties are also observed for all the experiments in two dimensions.

For some $\epsilon_M > 0$, $\epsilon_v > 0$, the density of $\mathcal{D}_n$ is small but the truncation error $\left\|\mathcal{D}_n - D_n^{\epsilon_M}\right\| < C_{\epsilon_M}$ and $\left\|u_n - u_n^{\epsilon_v}\right| < C_{\epsilon_v}$ is bounded for large n. Figure 7 gives an indication of the error committed by truncation of the solution. In all the schemes,

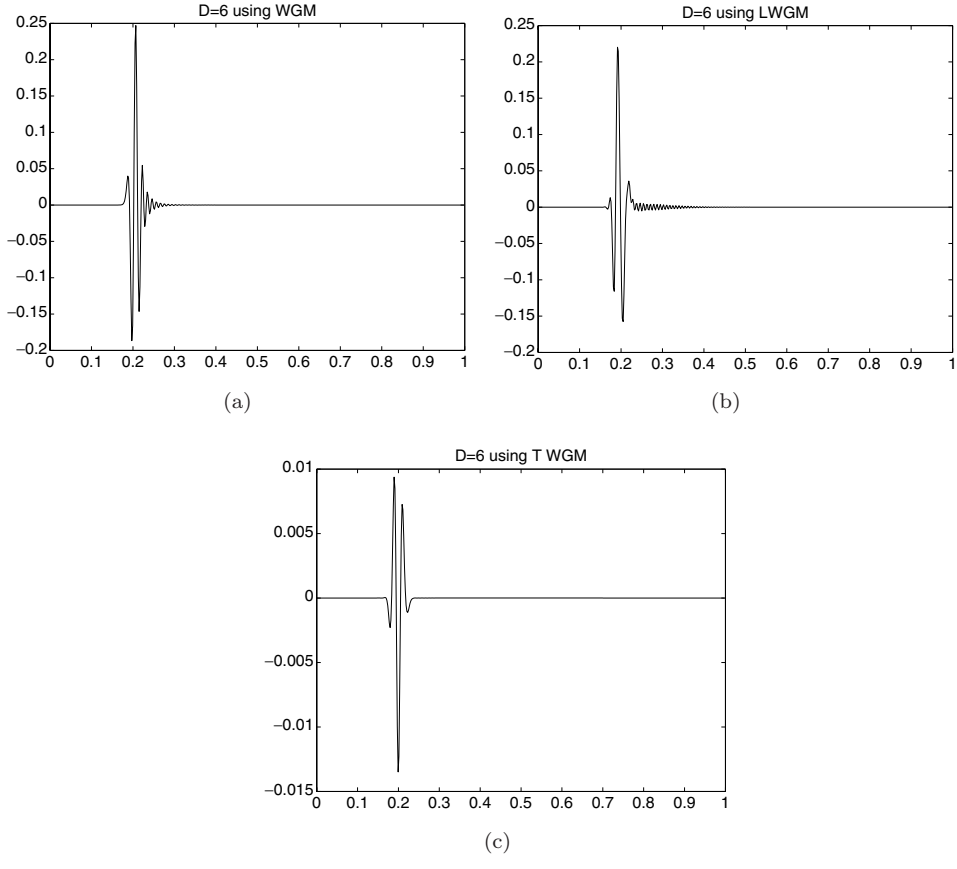

Fig. 6. Comparison of errors in approximated solution.

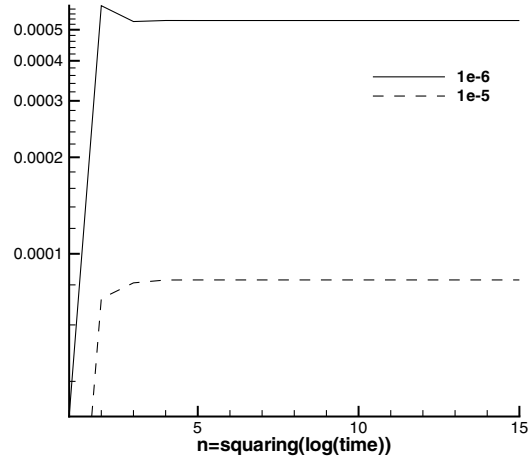

Fig. 7. Truncation errors $\|u_n - u_n^{\epsilon_v}\|$ for different threshold ϵ_v.

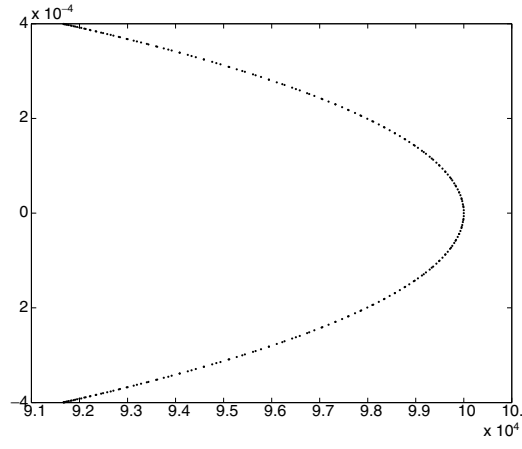

Fig. 8. For T-WGM, δt times the eigenvalues of $\mathcal{D}$ for $j = 8$.

δt is chosen such that it should satisfy stability condition. The $\lambda\delta t$ for the T-WGM scheme is plotted in Fig. 8.

3.2. *T-WGM for 2D advection equation*

As in the one-dimensional case, we first examine a linear advection equation. Again it will test the method's ability to follow features of the solution:

$$u_t = u_x + u_y. \tag{3.7}$$

Here, we are using the third-order accurate T-WGM scheme based on (3.6). Let $u_t = \nabla u$ and putting this values in Eq. (3.6):

$$\begin{aligned} u^{n+1/3} &= u^n + D_1 u^n, \\ u^{n+1/2} &= u^n + D_2 u^{n+1/3}, \\ u^{n+1} &= u^n + D_3 u^{n+1/2}, \end{aligned} \tag{3.8}$$

where $D_1 = \frac{\delta t}{3} a\nabla$, $D_2 = \frac{\delta t}{2} a\nabla$ and $D_3 = \delta t a\nabla$. All the algorithm in two dimensions will also take the advantage of wavelet compression like the one-dimensional case.

3.2.1. *Numerical simulation of 2D advection equation*

The initial function is the smooth function with a localized spike $u_0(x, y) = \exp^{-\alpha((x-1/2)^2+(y-1/2)^2)} - 0.2\sin(2\pi x)\sin(2\pi y)$ and periodic boundary conditions. In Fig. 9, the inital function and solution at $t = 0.3$ without and with truncation is shown for $\alpha = 2 \times 10^4$.

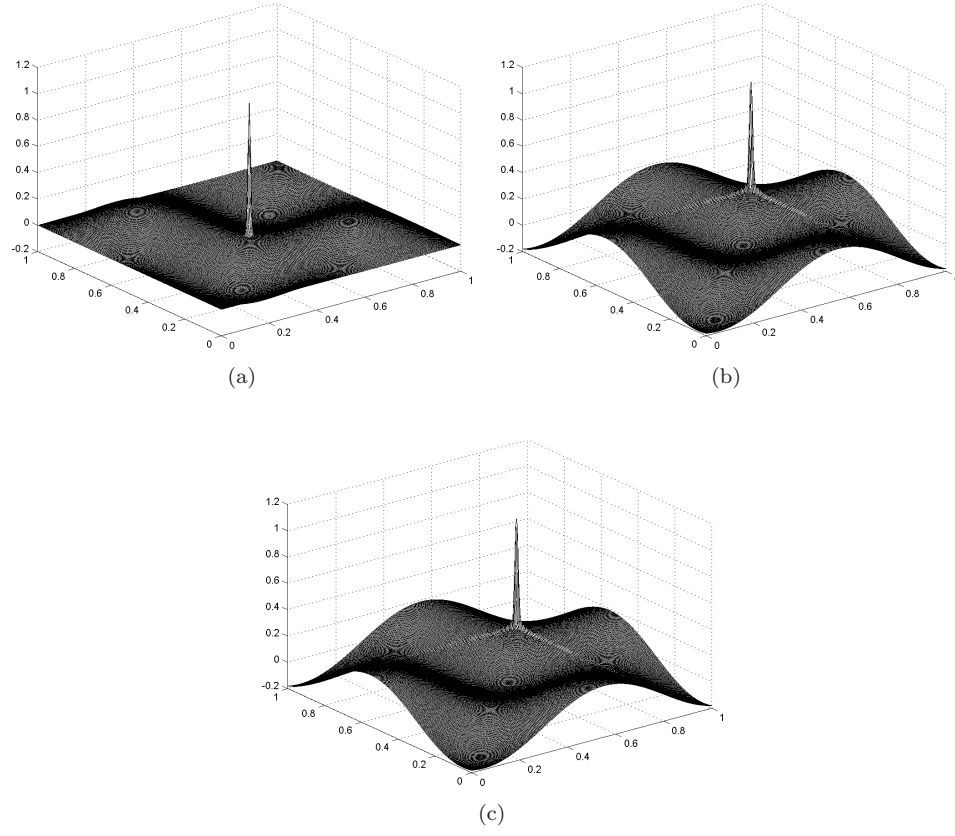

Fig. 9. (a) The initial function; (b) Solution at $t = 0.3$ when $j = 8, \epsilon_M = 10^{-5}, \epsilon_v = 0$; (c) Solution at $t = 0.125$ for $\epsilon_M = 10^{-5}$, $\epsilon_v = 10^{-4}$.

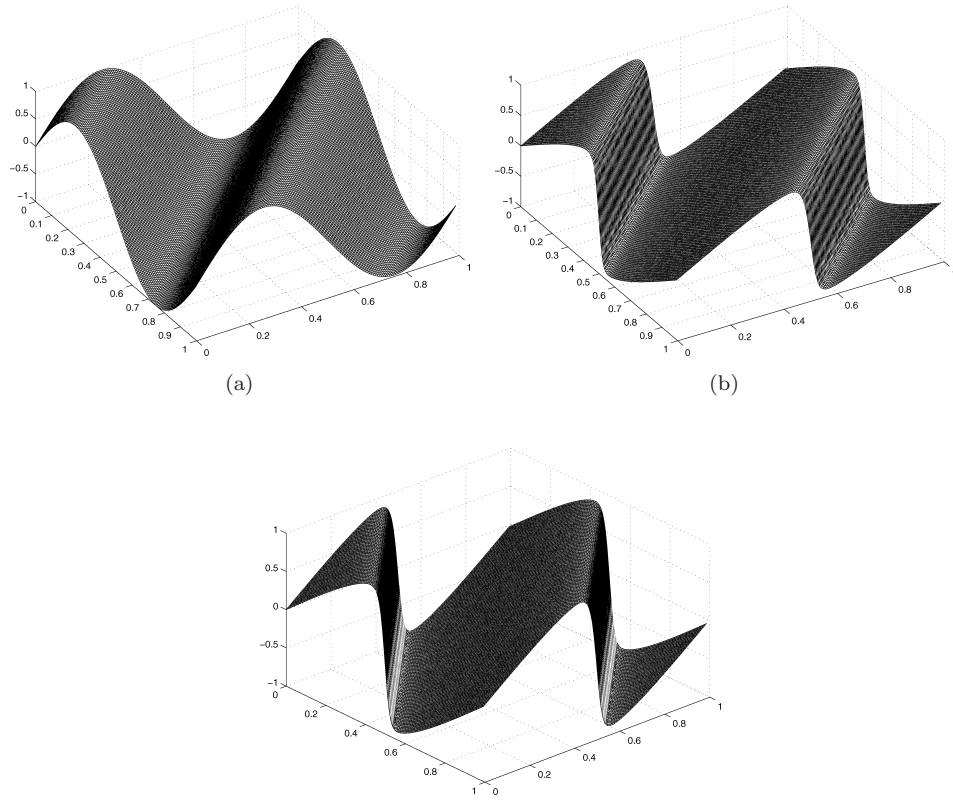

Fig. 10. (a) The initial function; (b) Solution at $t = 0.125$ when $j = 7$, $\epsilon_M = 10^{-5}$, $\epsilon_v = 0$; (c) Solution at $t = 0.125$ for $\epsilon_M = 10^{-5}$, $\epsilon_v = 10^{-4}$.

3.3. *T-WGM for 2D nonlinear equation*

Now, we would like to examine the ability of our method in two-dimensions when the gradient has developed in the solution. We examine the following two-dimensional counterpart to the one-dimensional Burgers' equation:

$$u_t + u(u_x + u_y) = \nu(u_{xx} + u_{yy}). \tag{3.9}$$

Here, we are using the third-order accurate T-WGM scheme based on (3.6). Let $f(x, y, u) = u(u_x + u_y)$, then the original equation is $u_t = \nu\Delta u - f(x, y, u)$ and putting this value in Eq. (3.6),

$$\begin{aligned} u^{n+1/3} &= u^n + D_1 u^n - (\delta t/3) f, \\ u^{n+1/2} &= u^n + D_2 u^{n+1/3} - (\delta t/2) f, \\ u^{n+1} &= u^n + D_3 u^{n+1/2} - \delta t f, \end{aligned} \tag{3.10}$$

where $D = \frac{\nu\delta t}{3}\Delta u$, $D = \frac{\nu\delta t}{2}\Delta u$ and $D = \nu\delta t\Delta u$.

3.3.1. *Numerical simulation of* 2*D nonlinear equation*

If we choose the initial function as a two-dimensional sine wave $u_0(x, y) = \sin(2\pi(x + y))$ and periodic boundary conditions, the gradient in the shock will reach its maximum at time $t = 0.125$ when the extrema of the sine wave have advected into the shock. In Fig. 10, the inital function and solution at $t = 0.125$ without and with truncation is shown for $\nu = 10^{-2}$.

4. Conclusion

In the three-step wavelet-Galerkin method, the precedence of time discretization to space discretization in conjunction with wavelet bases for expressing spatial terms renders robustness to the proposed schemes and makes them space and time accurate. The three-step wavelet-Galerkin method retains the third-order accuracy and stability property of the Taylor–Galerkin method. The concepts are introduced through linear advection equation and Burgers' equation where we can show the power of wavelet compression as well as time accurate schemes. Since no new higher-order derivative term occurs in the numerical formulations, the present method is suitable for nonlinear problems. Further, the method can be directly extended to three-dimensional problems. The numerical results show that the present method is computationally efficient.

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