International Journal of Wavelets, Multiresolution
and Information Processing
Vol. 5, No. 4 (2007) 667–678

ERROR ESTIMATES FOR TIME ACCURATE WAVELET BASED SCHEMES FOR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

MANI MEHRA* and B. V. RATHISH KUMAR†

Department of Mathematics, Indian Institute of Technology
Kanpur, U. P., 208016, India
**manimeh2001@yahoo.co.in*
†bvrk@iitk.ac.in

Received 9 June 2006
Revised 3 December 2006

In this study, we derive error estimates for time accurate wavelet based schemes in two stages. First we look at the semi-discrete boundary value problem as a Cauchy problem and use spectral decomposition of self adjoint operators to arrive at the temporal error estimates both in L^2 and energy norms. Later, following the wavelet approximation theory, we propose spatial error estimates in L^2 and energy norms. And finally arrive at *a priori* estimates for the fully discrete problem.

Keywords: Multiresolution analysis; wavelets; time accurate schemes; hyperbolic equation.

Mathematics Subject Classification 2000: 65M12, 65M15, 65M70

1. Introduction

The application of methods based on wavelets to the numerical solution of partial differential equations (PDEs) has recently been studied both from the theoretical and the computational points of view due to its attractive feature: orthogonality, arbitrary regularity, and good localization. Wavelet bases seem to combine the advantages of both spectral and finite element bases. Wavelet elements have the potential of capturing localized phenomena and computing multiscale solution to partial differential equations with higher convergence rates than conventional finite element methods. In the finite element solution of convection dominated problems exhibiting internal or boundary layers, advantage can be taken of the properties of wavelet-based approximation to achieve an accurate localization of such layers. Dahmen *et al.*[1] provide an overview of recent progress in the development and use of wavelet methods in the fluid mechanics area.

†Corresponding author.

In the conventional numerical approach to transient problems the accuracy gained in using the high order spatial discretization is partially lost due to use of low-order time discretization schemes. Here, usually spatial approximation precedes the temporal discretization. On the contrary, the reversed order of discretization can lead to better time accurate schemes with improved stability properties.[2] The fundamental concept behind the Taylor Galerkin (TG) schemes (time accurate schemes) is to incorporate more analytic information in to numerical scheme in the most direct and natural way, so that the technique may be regarded as an extension to PDEs of the obrechkoff methods for ODE. Safjan and Oden have derived a new family of high order Taylor Galerkin scheme which are unconditionally stable.[3]

Numerically, the high order convergence rates and asymptotic analysis of a new class of wavelet based TG schemes have been established by the authors in Refs. 4 and 5. Time accurate fast wavelet TG schemes is also developed[6] where we use the compression properties of wavelet bases. These TG schemes also have excellent accuracy; some of them posses a built-in temporal error estimate. In this paper, we derive *a priori* error estimates of time accurate wavelet based schemes for hyperbolic problems. The main conceptual ingredients center around the spectral decomposition of self adjoint operator and wavelet approximation theory.

The contents of the paper is organized as follows. In Sec. 2, we summarize some basics of wavelet analysis. In Sec. 3, we consider the abstract formulation of the problem and discretization of problem. In Sec. 4, we present wavelet error estimation for time accurate schemes.

2. General Setting

Definition 1. A multiresolution analysis (MRA) of $L_2(\mathbb{R}^\nu)$ describes a sequence of nested approximation spaces $V_j, j \in \mathbb{Z}$ such that closure of their union equals $L^2(\mathbb{R}^\nu)$. MRA is characterized by the following axioms

$$\begin{aligned}
&\{0\} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \cdots \subset L^2(\mathbb{R}^\nu),\\
&\overline{\bigcup_{j=-\infty}^{j=\infty}} V_j = L^2(\mathbb{R}^\nu),\\
&\bigcap_{j\in Z} V_j = 0, \\
&f \in V_j \text{ if and only if } f(2(.)) \in V_{j+1},\\
&\phi(x-k)_{k\in\mathbb{Z}} \text{ is an orthonormal basis for } V_0.
\end{aligned} \tag{2.1}$$

We define W_j to be the orthogonal complement of V_j in V_{j+1}, i.e. $V_j \perp W_j$ and

$$V_{j+1} = V_j + W_j. \tag{2.2}$$

There exists a function, which is called a scaling function $\phi(x) \in V_0$, such that the sequence $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)_{k\in\mathbb{Z}}$ is an orthonormal basis for V_j and

$\psi_{j,k} = 2^{j/2}\psi(2^j x - k)_{k\in\mathbb{Z}}$ is an orthonormal basis for W_j. Mathematically, one introduces at each step j, the subspace W_j, defined as the orthogonal complement of V_j in V_{j+1}. Then one has the fundamental theorem proved by Mallat[7] and Meyer.[8]

Theorem 2.1. *There exists a function of* W_0 *such that* $\psi(x-k)$, $k \in \mathbb{Z}$ *is an orthonormal basis of* W_0. ψ *has the regularity properties*

$$\int x^k \psi(x)dx = 0 \quad \textit{for } 0 \le k \le D/2 - 1. \tag{2.3}$$

As pointed out by Meyer the complete toll box built in $L^2(\mathbb{R})$ can be used in the periodic case $L^2([0,1])$ by introducing a standard periodization technique. This technique consists at each scale in folding, around the integer values, the wavelet $\psi_{j,k}$ and the scaling functions $\phi_{j,k}$ centered in [0, 1].

We indicate $\lambda = (j,k)$ an index in Z^2. By Λ we will indicate the set of all admissible indexes λ, and we can write any distribution $u \in H^{-1}(0,1)$ as $u = \mathcal{P}_{j_0}u + \sum_{\lambda\in\Lambda}\langle u, \psi_\lambda\rangle\psi_\lambda$ satisfying $j \ge j_0$. The approximation space $\mathcal{V}_\mathcal{N}$ will be constructed by choosing a subset of indexes $\Lambda_\mathcal{N} \subset \Lambda$ and writing

$$\mathcal{V}_\mathcal{N} = V_{j_0} \cup \operatorname{span}\langle\psi_\lambda, \lambda \in \Lambda_\mathcal{N}\rangle.$$

Simple thresholding of the largest contribution in the wavelet composition provides compressed solution u_h such that $u_h = \mathcal{P}_h u$, where $\mathcal{P}_h$ is the projection defined on the space $\mathcal{V}_\mathcal{N} = \sum_{\lambda\in\Lambda_\mathcal{N}} u_\lambda\psi_\lambda$. We set $\mathcal{V}_\mathcal{N} = \mathcal{V}_h$ with $\mathcal{N} = h^{-\nu}$, $h = 2^{-j}$.

3. Higher-Order Taylor Galerkin Methods

Within the Hilbert space formulation, the initial boundary value problem can be reinterpreted as an abstract Cauchy problem for operator $\mathcal{A}$.

$$\begin{aligned} &\frac{d}{dt}u + \mathcal{A}u = 0, && 0 < t \le t^*, \\ &u(0) = u^0, && t = 0. \end{aligned} \tag{3.1}$$

For a variational formulation of this problem we introduce Sobolev spaces. Let $\Omega \subset \mathbb{R}^\nu$ be a bounded domain with periodic boundary $\Gamma = \partial\Omega$. We denote by $H = L^2(\Omega)$ the usual square integrable functions with inner product $(.,.)$ and by $H^s(\Omega), s \ge 0$, the corresponding Sobolev spaces.[9] We assume that $\mathcal{A} \in \mathcal{L}(V; V^*)$. By $(.,.)_{V^*\times V}$ we denote the extension of $(.,.)$ as duality pairing in $V^* \times V$, and by $\|.\|$, $\|.\|_V$, $\|.\|_{V^*}$ the norms in $L^2(\Omega)$, V, V^* respectively and $\|.\|_\mathcal{A}$ denote the graph norm $(\|.\|^2 + \|\mathcal{A}.\|^2)^{1/2}$. Then $D(\mathcal{A}) \subset V \subset V^*$. We associate with $\mathcal{A}$ the bilinear form $a(.,.) : V \times V \to \mathbb{C}$ via

$$a(u,v) = (\mathcal{A}u, v)_{V^*\times V}, \quad u, v \in V. \tag{3.2}$$

Then the form $a(.,.)$ is continuous,

$$|a(u,v)| \le \alpha\|u\|_V\|v\|_V, \quad \forall u, v \in V \tag{3.3}$$

and we assume that it is coercive in the sense that

$$a(u, v) \geq \beta \|u\|_V^2 \tag{3.4}$$

for some $0 < \beta \leq \alpha < \infty$. Then $\mathcal{A} \in \mathcal{L}(V, V^*)$ is an isomorphism and $\|\mathcal{A}\|_{\mathcal{L}(V,V^*)} \leq \alpha$. The time derivative $\dot{u}(t)$ in (3.1) is understood in the weak sense; i.e., for $u \in L^2((0, t^*); V)$ we have $\dot{u} \in L^2((0, t^*); V^*)$ defined by

$$\int_J (\dot{u}(t), v)_{V^* \times V} \varphi(t) dt = -\int_J (u(t), v) \dot{\varphi}(t) dt \tag{3.5}$$

for every $v \in V$, $\varphi \in C_0^\infty((0, t^*))$. If the initial condition satisfies an additional regularity assumption $u^0 \in D(\mathcal{A})$ then the solution $u \in C^1((0, \infty); V) \cap C((0, \infty); V^*)$. We say ten that u is a strict solution to the problem.

We record now some fundamental results concerning operator $\mathcal{A}$ and the existence and uniqueness of weak solution u:

(i) Operator $\mathcal{A}$ is self adjoint and, therefore, its spectrum lies in the real line and consists of a point spectrum and continuous spectrum. For bounded domain, the spectrum of $\mathcal{A}$ consists of eigenvalues only. Except for the 0-eigenvalue, all the eigenvalues are of finite multiplicity and the corresponding eigenspaces $\{u_n\}$ are orthogonal.

(ii) Operator $\mathcal{A}$ admits a classical spectral decomposition

$$\mathcal{A}u = \int_{-\infty}^{\infty} \lambda dE_\lambda u, \tag{3.6}$$

$$D(\mathcal{A}) = \left\{ u \in H : \int_{-\infty}^{\infty} \lambda^2 d\|E_\lambda u\|^2 < \infty \right\}, \tag{3.7}$$

where E_λ is a uniquely defined spectral family of $\mathcal{A}$.[9]

(iii) A real solution u exists and is unique. Moreover, it is of the form

$$u(t) = e^{-\mathcal{A}t} u^0 = \int_{-\infty}^{\infty} e^{-\lambda t} dE_\lambda u^0. \tag{3.8}$$

In particular, it follows from (3.8) that the energy is conserved.

$$\|u(t)\|^2 = \int_{-\infty}^{\infty} |e^{-t\lambda}|^2 d(E_\lambda u^0, u^0) = \|u^0\|^2, \quad \forall t \geq 0 \tag{3.9}$$

3.1. *Approximation in time*

By discretizing in time first, the initial boundary value problem (3.1) is converted into a sequence of boundary value problems (3.10),

$$\begin{aligned} u_\tau(x, t_n + \delta t) &= \mathcal{T} u_\tau(x, t_n), \\ u_\tau(x, 0) &= u^0, \\ \delta t = t^*/N, \quad t_n &= n\delta t, \quad n = 0, 1, \ldots, N. \end{aligned} \tag{3.10}$$

The low order time schemes do not allow a sufficiently accurate approximation of the exponential operator in (3.8), or, stated in other words, they do not properly account for the directional character of propagation of information in hyperbolic problems. Higher order time stepping schemes provide a better approximation to the exponential function in (3.8), and consequently allow a better account of the propagation of information along the characteristics. Such methods represent an attempt to simulate, by a Taylor series in time extended to third and fourth order.

Third order Taylor Galerkin scheme (E-TGS) Taylor Galerkin method represent a generalization of the Lax–Wendroff method. They are based on a Euler-Taylor series expansion up to desired order. Thus the solution u must be sufficiently smooth. In order to obtain a third order method the Taylor series is taken as

$$\frac{u^{n+1}-u^n}{\delta t} = u_t^n + \frac{\delta t}{2}u_{tt}^n + \frac{\delta t^2}{6}u_{ttt}^n + O(\delta t^3). \tag{3.11}$$

Three step Taylor Galerkin schemes (T-TGS) A three-step finite element method based on a Taylor series expansion in time is proposed in Ref. 10. This scheme involves neither complicated expression nor higher order derivatives like in E-TGS. By approximating Eq. (3.11) up to third order accuracy, the formulations of the three-step method can be written as

$$\begin{aligned} u\left(t+\frac{\delta t}{3}\right) &= u(t) + \frac{\delta t}{3}\frac{\partial u(t)}{\partial t}, \\ u\left(t+\frac{\delta t}{2}\right) &= u(t) + \frac{\delta t}{2}\frac{\partial u(t+\delta t/3)}{\partial t}, \\ u(t+\delta t) &= u(t) + \delta t\frac{\partial u(t+\delta t/2)}{\partial t}. \end{aligned} \tag{3.12}$$

3.2. *Approximation in space*

First, using the original equations (3.1), we calculate the time derivatives in terms of spatial derivatives as follows

$$u_t = -\mathcal{A}u, \quad u_{tt} = \mathcal{A}^2 u, \quad u_{ttt} = \mathcal{A}^2 u_t. \tag{3.13}$$

We consider E-TGS in the illustration of above procedure. Consider $\mathcal{V}_\mathcal{N}^p \subset V$ of periodic version of space $\mathcal{V}_\mathcal{N}$ and $\mathbb{V}_N^p = \mathcal{V}_\mathcal{N}^p \cap D(\mathcal{A})$. To discretization in space we use wavelet projection $\mathcal{P}_h^c : V \to \mathbb{V}_N^p$. Now replacing the time derivatives in (3.11) and multiplying by a test function $v_h \in \mathbb{V}_N^p$, integrating over Ω, and integrating by parts, we arrive at a variational formulation of the form,

$$\begin{aligned} &\text{given } u_h^0 \in \mathbb{V}_N^p, \\ &\mathcal{B}(u_h^{n+1}, v_h) + \frac{\delta t^2}{6}\mathcal{C}(u_h^{n+1}, v_h) = \mathcal{B}(u_h^n, v_h) + \delta t\mathcal{D}(u_h^n, v_h) - \frac{\delta t^2}{3}\mathcal{C}(u_h^n, v_h), \end{aligned} \tag{3.14}$$

where the bilinear forms $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are defined by

$$\begin{aligned} &\mathcal{B},\mathcal{C},\mathcal{D} : V \times V \to \mathbb{C} \\ &\qquad \mathcal{B}(u,v) = (u,v), \quad \mathcal{C}(u,v) = (\mathcal{A}u, \mathcal{A}v), \quad \mathcal{D}(u,v) = (\mathcal{A}u, v), \end{aligned} \tag{3.15}$$

and $\mathcal{B}_1(.,.)$ is defined by

$$\mathcal{B}_1(u,v) = \mathcal{B}(u,v) + \frac{\delta t^2}{6}\mathcal{C}(u,v) \quad \forall u, \quad v \in V.$$

It is easy to check that the bilinear form $\mathcal{B}_1$ is continuous and coercive with respect to $\|.\|_{\mathcal{A}}$ norm. And therefore, by the virtue of the Lax–Milgram theorem, the bilinear form is well defined.

4. *A Priori* Error Estimation for Time Accurate Schemes

4.1. *Temporal approximation error*

Let $u(t)$ be the solution of the Cauchy problem (3.1) and let $u_\tau(t)$ be its semidiscrete approximation as in (3.10). To estimate the error in TG schemes, we need the following lemmas.

Lemma 4.1. *If $\mathcal{T}$ is a transient operator corresponding to different TG schemes. then $\mathcal{T}$ can be represented as a rational function of the underlying operator $\mathcal{A}$,*

$$\mathcal{T}u = r(\delta t\mathcal{A})u. \tag{4.1}$$

Proof. We prove the above assertion for E-TGS scheme. For the E-TGS scheme

$$\begin{aligned} \mathcal{T}u &= \left(\mathcal{I} + \frac{\delta t^2}{6}\mathcal{A}^2\right)^{-1}\left[I + \mathcal{A}\delta t - \frac{\delta t^2}{3}\mathcal{A}^2 u\right] \\ &= \int_{-\infty}^{\infty} r(\delta t\lambda)dE_\lambda u = r(\delta t\mathcal{A})u, \end{aligned}$$

where $r : \mathbb{Z} \to \mathbb{Z}$ is given by

$$\begin{aligned} r(z) &= (1 + \frac{z^2}{6})^{-1}\left(1 + z - \frac{z^2}{3}\right), \\ r(\mathbb{Z}) &= P_1(z)/(1 + z^2/6). \end{aligned}$$

P_1 is are second order polynomial. Hence (4.1) follows. □

Remark 4.2. If $\mathcal{T}$ is a transient operator corresponding to T-TGS scheme. Then $r(\mathbb{Z}) = P_2(z)$ for T-TGS scheme where P_2 is also second order polynomial.

Lemma 4.3. *If $\mathcal{T}$ is a transient operator corresponding to different TG schemes, then for an mth-order TG scheme,*

$$|r(y) - e^{-y}| = O(|y^{m+1}|), \quad y \in \mathbb{R}. \tag{4.2}$$

Proof. It directly follows from the definition. □

Remark 4.4. We use the notion of asymptotic stability of a numerical method for a discrete problem of the form $du/dt = \mathcal{L}u$ where $\mathcal{L}$ is assumed to be diagonalization matrix. The region of absolute stability of a numerical method is defined for the scalar model problem $du/dt = \lambda u$ to be set of all $\lambda\delta t$ such that $||u^n||$ is bounded as $t \to \infty$. Finally we say that a numerical method is asymptotically stable for a particular problem if, for small $\delta t > 0$, the product of δt times every eigenvalues of $\mathcal{L}$ lies within the region of absolute stability. The region of absolute stability for wavelet-Taylor Galerkin schemes are defined in Ref. 4.

Lemma 4.5. *If $\mathcal{T}$ is a transient operator corresponding to different TG schemes then, for asymptotically stable scheme there exists a δt such that*

$$|r(\delta t\lambda)| \leq 1. \tag{4.3}$$

We now define E_τ, the temporal approximation error, $E_\tau = u(t^*) - u_\tau(t^*)$. To estimate $\|E_\tau\|$ we need the following result.

Lemma 4.6. *Let $u \in D(\mathcal{A}^{m+1})$, $\mathcal{T}$ be a transient operator corresponding to different TG schemes and* (4.2) *holds. Then there exists a constant c such that the one step error satisfies*

$$\|e^{-\mathcal{A}\delta t}u - r(\delta t\mathcal{A})u\| \leq c\delta t^{m+1}\|\mathcal{A}^{m+1}u\|. \tag{4.4}$$

Proof.

$$\begin{aligned}
\|e^{-\mathcal{A}\delta t}u - r(\delta t\mathcal{A})u\|^2 &= \int_{-\infty}^{\infty} |e^{-\lambda\delta t} - r(\lambda\delta t)|^2 d(E_\lambda u, u) \\
&\leq (c\delta t^{m+1})^2 \int_{-\infty}^{\infty} |\lambda^{m+1}|^2 d(E_\lambda u, u) \\
&= (c\delta t^{m+1})^2 \|\mathcal{A}^{m+1}u\|^2
\end{aligned}$$

□

The main consequence of the above lemmas can be formulated as the follows.

Lemma 4.7. *Let $\mathcal{T}$ be a transient operator corresponding to different TGS schemes and* (4.2) *and* (4.3) *holds then the error estimate is bounded by*

$$\|E_\tau\| \leq ct^*\delta t^m \|\mathcal{A}^{m+1}u^0\| \tag{4.5}$$

for all $u^0 \in D(\mathcal{A}^{m+1})$.

Proof.

$$\begin{aligned}
\|E_\tau\| &= \|e^{-N\mathcal{A}\delta t}u^0 - r^N(\delta t\mathcal{A})u^0\| \\
&\leq \sum_{j=0}^{N-1} \|e^{-(N-j-1)\delta t\mathcal{A}}\| \|r^j(\delta t\mathcal{A})\| \|e^{-\mathcal{A}\delta t}u^0 - r(\delta t\mathcal{A})u^0\| \\
&\leq N\|e^{-\mathcal{A}\delta t}u^0 - r(\delta t\mathcal{A})u^0\| \quad \text{(using Lemma (4.6))}
\end{aligned}$$

$$\leq Nc\delta t^{m+1}\|\mathcal{A}^{m+1}u^0\|$$
$$= ct^*\delta t^m\|\mathcal{A}^{m+1}u^0\|, \quad u^0 \in D(\mathcal{A}^{m+1}),$$

which completes the proof. □

We also estimate temporal approximation error in the energy norm $\|.\|_E$ defined by the bilinear form $\|.\|_E = \mathcal{B}_1(.,.)^{1/2}$ where for E-TGM scheme $\mathcal{B}_1(.,.) = \mathcal{B}(.,.) + \frac{\delta t^2}{6}\mathcal{C}(.,.)$.

Corollary 4.8. *Let $\mathcal{T}$ be a transient operator corresponding to different TGS schemes and* (4.2) *and* (4.3) *holds then,*

$$\|E_\tau\|_E \leq ct^*\delta t^m\|\mathcal{A}^{m+1}u^0\|_E \tag{4.6}$$

for all $u^0 \in D(\mathcal{A}^{m+2})$.

Proof. Assume $u \in D(\mathcal{A}^{m+2})$ then one step error is estimated by

$$\begin{aligned}\|e^{-\mathcal{A}\delta t}u - r(\delta t\mathcal{A})u\|_E^2 &= \int_{-\infty}^{\infty} |e^{-\lambda\delta t} - r(\lambda\delta t)|^2(1+\delta t^2/6)d(E_\lambda u, u)\\ &\leq (c\delta t^{m+1})^2\int_{-\infty}^{\infty}|\lambda^{m+1}|^2(1+\delta t^2/6)d(E_\lambda u, u)\\ &= (c\delta t^{m+1})^2\|\mathcal{A}^{m+1}u\|_E^2\end{aligned}$$

Next, we note that

$$\|e^{-\delta t\mathcal{A}}\|_E^2 = \|u\|_E^2, \quad \|r(\delta t\mathcal{A})u\|_E^2 \leq \|u\|_E^2$$

Consequently, assuming $u^0 \in D(\mathcal{A}^{m+2})$, the estimate (4.5) is also valid in the energy norm. □

4.2. *Spatial approximation error*

We now consider the spatial approximation of (3.10) by wavelet Galerkin method

$$\begin{aligned} u_{\tau h}(x, t_n + \delta t) &= \mathcal{T}_h u_{\tau h}(x, t_n),\\ u_{\tau h}(x, 0) &= u^0 \\ \delta t = t^*/N, \quad t_n &= n\delta t, \quad n = 0, 1, \ldots, N.\end{aligned} \tag{4.7}$$

We now define E_h, the spatial approximation error, $E_h = u_\tau(t^*) - u_{\tau h}(t^*)$ and we estimate it in energy norm. The rate of convergence of wavelet scheme to the exact solution with respect to the number $\mathcal{N}$ of degrees of freedom, i.e. of wavelets which are used to describe the solution is the same as the rate of convergence of the best $\mathcal{N}$-term approximation, which would be obtained by retaining the $\mathcal{N}$ largest wavelet coefficients of the exact solution. Here we use the following approximation property of the wavelet.

Definition 2. A MRA of $L^2(\mathbb{R}^\nu)$ is said to be r-regular ($r \in \mathbb{N}$) if the function ϕ is r-regular, that is, for each $m \in \mathbb{N}$ there exists c_m such that for all multi-indexes α, $|\alpha| \leq r$, the following conditions holds: $|D^\alpha \phi(x)| \leq c_m(1+|x|)^{-m}$.
Now we have the following wavelet approximation theorem.[11]

Theorem 4.9. *Let $r-1 \leq q \leq r$, $-r \leq s \leq r+1$ and $q \leq s$. Then*

$$\|u - \mathcal{P}_h u\|_{H^q(\Omega)} \leq \tilde{c} h^{s-q} \|u\|_{H^s(\Omega)} \tag{4.8}$$

for all $u \in H^s(\Omega)$, where $\tilde{c}$ is a constant independent of h and u.

This lemma shows that a smoothness property implies good approximation property of wavelet projection operator. Now we first record the following inequality using Theorem (4.9).

Lemma 4.10. *Let $\mathcal{T}$ be a transient operator corresponding to different TG schemes, then*

$$\|\mathcal{P}_h \mathcal{T} u - \mathcal{T}_h \mathcal{P}_h u\| \leq c^* h^s \|u\|_{H^s(\Omega)} \tag{4.9}$$

for all $u \in H^s(\Omega)$, where c^* is a constant independent of h and u.

Proof.

$$\begin{aligned} \|\mathcal{P}_h \mathcal{T} u - \mathcal{T}_h \mathcal{P}_h u\| &= \|\mathcal{P}_h \mathcal{T} u - \mathcal{P}_h \mathcal{T} \mathcal{P}_h u\| \\ &\leq \|\mathcal{P}_h\| \|\mathcal{T} u - \mathcal{T} \mathcal{P}_h u\| \\ &\leq \|\mathcal{T} u - \mathcal{T} \mathcal{P}_h u\| \\ &\leq \|\mathcal{T}\| \|u - \mathcal{P}_h u\| \\ &\leq \|\mathcal{T}\| \|u - \mathcal{P}_h u\| \\ &\leq M \|u - \mathcal{P}_h u\| \\ &\leq c^* h^s \|u\|_{H^s(\Omega)}. \end{aligned}$$

□

We note the uniform quasi-boundedness of $\mathcal{T}_h^N$ by

$$\|\mathcal{T}_h^N\| = \sup_{u \in \mathcal{V}_h, u \neq 0} \frac{\|\mathcal{T}_h^N u\|}{\|u\|} \leq L e^{Bt^*} \text{ independent of } N.$$

Now, using the above lemmas, the spatial approximation error $\|E_h\|$ can be bounded as follows

Lemma 4.11. *Let $\mathcal{T}$ be a transient operator corresponding to different TG schemes, then*

$$\|E_h\| \leq N L e^{Bt^*} c^* h^s \|\mathcal{T}^{I-1} u^0\|_{H^s(\Omega)} + c^* h^s \|\mathcal{T}^N u^0\|_{H^s(\Omega)} \tag{4.10}$$

for all $u^0 \in H^s(\Omega)$.

Proof.

$$\begin{aligned}\|E_h\| &= \|\mathcal{T}^N u^0 - \mathcal{T}_h^N \mathcal{P}_h u^0\| \\ &\le \sum_{i=1}^{i=N} \|\mathcal{T}_h^{N-i}\|\|(\mathcal{P}_h\mathcal{T} - \mathcal{T}_h\mathcal{P}_h)\mathcal{T}^{i-1}u^0\| + \|\mathcal{T}^N u^0 - \mathcal{P}_h\mathcal{T}^N u^0\| \\ &\le NLe^{Bt^*}\|(\mathcal{P}_h\mathcal{T} - \mathcal{T}_h\mathcal{P}_h)\mathcal{T}^{I-1}u^0\| + \|\mathcal{T}^N u^0 - \mathcal{P}_h T^N u^0\| \\ &\le NLe^{Bt^*}c^* h^s \|\mathcal{T}^{I-1}u^0\|_{H^s(\Omega)} + c^* h^s \|\mathcal{T}^N u^0\|_{H^s(\Omega)}, \quad \forall u^0 \in H^s(\Omega),\end{aligned} \tag{4.11}$$

where

$$\|(\mathcal{P}_h\mathcal{T} - \mathcal{T}_h\mathcal{P}_h)\mathcal{T}^{I-1}u^0\| = \max_{i=1,2,\ldots,N} \|\mathcal{P}_h\mathcal{T} - \mathcal{T}_h\mathcal{P}_h)\mathcal{T}^{i-1}u^0\| \qquad \square$$

Now we will prove same wavelet error estimate for energy norm in the subsequent lemmas. First, we are proving approximation error of wavelet projection in the energy norm as follows

Lemma 4.12. *Let $r-1 \le q \le r$, $-r \le s \le r+1$ and $1 \le s \ge q$. Then*

$$\|u - \mathcal{P}_h u\|_E \le \check{c} h^s \|u\|_{H^s(\Omega)}$$

for all $u \in D(\mathcal{A}) \cap H^s(\Omega)$, where $\check{c}$ is a constant independent of h and u.

Proof.

$$\begin{aligned}\|u - \mathcal{P}_h u\|_E &= \inf_{\chi \in V_h} \{\|u - \chi\|_E\} \\ &= \inf_{\chi \in V_h} \{(\|u-\chi\|^2 + (\delta t^2/6)\|\mathcal{A}(u-\chi)\|^2)^{1/2}\} \text{ for E-TGS scheme} \\ &\le \inf_{\chi \in V_h} \{\|u-\chi\| + (\delta t/\sqrt{(}6))\|A(u-\chi)\|\} \\ &\le \check{c} h^s \|u\|_{H^s(\Omega)} \forall u \in D(\mathcal{A}) \cap H^s(\Omega) \text{ using (4.8).}\end{aligned} \tag{4.12}$$

$\square$

Now we first record the following inequality using Lemma 4.12.

Lemma 4.13. *Let $\mathcal{T}$ be a transient operator corresponding to different TG schemes, then*

$$\|\mathcal{P}_h\mathcal{T}u - \mathcal{T}_h\mathcal{P}_h u\|_E \le \check{c} h^s \|u\|_{H^s(\Omega)} \tag{4.13}$$

for all $u \in D(\mathcal{A}) \cap H^s(\Omega)$, $s \ge 1$ where $\check{c}$ is a constant independent of h and u.

Proof.

$$\begin{aligned}\|\mathcal{P}_h\mathcal{T}u - \mathcal{T}_h\mathcal{P}_h u\|_E &= \|\mathcal{P}_h\mathcal{T}u - \mathcal{P}_h\mathcal{T}\mathcal{P}_h u\|_E \\ &\le \|\mathcal{P}_h\|_E \|\mathcal{T}u - \mathcal{T}\mathcal{P}_h u\|_E \\ &\le \|\mathcal{T}u - \mathcal{T}\mathcal{P}_h u\|_E\end{aligned}$$

$$\begin{aligned}
&\leq \|\mathcal{T}\|_E \|u - \mathcal{P}_h u\|_E \\
&\leq \|\mathcal{T}\|_E \|u - \mathcal{P}_h u\|_E \\
&\leq M \|u - \mathcal{P}_h u\|_E \\
&\leq \check{c} h^s \|u\|_{H^s(\Omega)}.
\end{aligned}$$ □

We note the uniform quasi-boundedness of $\mathcal{T}_h^N$ is also true in the energy norm, then the spatial approximation error $\|E_h\|_E$ is bounded as follows.

Lemma 4.14. *Let $\mathcal{T}$ be a transient operator corresponding to different TG schemes, then*

$$\|E_h\|_E \leq NLe^{Bt^*} \check{c} h^s \|\mathcal{T}^{I-1} u^0\|_{H^s(\Omega)} + \check{c} h^s \|\mathcal{T}^N u^0\|_{H^s(\Omega)} \tag{4.14}$$

for all $u^0 \in D(\mathcal{A}) \cap H^s(\Omega)$, $s \geq 2$.

Proof.

$$\begin{aligned}
\|E_h\|_E &= \|\mathcal{T}^N u^0 - \mathcal{T}_h^N \mathcal{P}_h u^0\|_E \\
&\leq \sum_{i=1}^{i=N} \|\mathcal{T}_h^{N-1}\|_E \|(\mathcal{P}_h \mathcal{T} - \mathcal{T}_h \mathcal{P}_h) \mathcal{T}^{i-1} u^0\|_E + \|\mathcal{T}^N u^0 - \mathcal{P}_h \mathcal{T}^N u^0\|_E \\
&\leq LNe^{Bt^*} \|\mathcal{P}_h \mathcal{T} - \mathcal{T}_h \mathcal{P}_h) \mathcal{T}^{I-1} u^0\|_E + \|\mathcal{T}^N u^0 - \mathcal{P}_h T^N u^0\|_E \\
&\leq LNe^{Bt^*} \check{c} h^s \|\mathcal{T}^{I-1} u^0\|_{H^s(\Omega)} + \check{c} h^s \|\mathcal{T}^N u^0\|_{H^s(\Omega)}
\end{aligned} \tag{4.15}$$

We are now in a position to prove the main theorem of this paper. □

Theorem 4.15. *Let $\mathcal{T}$ be a transient operator corresponding to different TG schemes, then the estimate of the total approximation error is bounded by*

$$\|E\| \leq f(t^*)[\delta t^m \|\mathcal{A}^{m+1} u^0\| + h^{s-1} \|u^0\|_{H^s(\Omega)}] \tag{4.16}$$

for all $u^0 \in D(\mathcal{A}^{m+1}) \cap H^s(\Omega)$.

Proof.

$$\begin{aligned}
\|E\| &= \|u(t^*) - u_{\tau h}(t^*)\| \\
&\leq \|E_\tau\| + \|E_h\| \\
&\leq ct^* \delta t^m \|\mathcal{A}^{m+1} u^0\| + NLe^{bt^*} \check{c} h^s \|\mathcal{T}^{I-1} u^0\|_{H^s(\Omega)} + \check{c} h^s \|\mathcal{T}^N u^0\|_{H^s(\Omega)} \\
&\leq f(t^*)[\delta t^m \|\mathcal{A}^{m+1} u^0\| + h^{s-1} \|u^0\|_{H^s(\Omega)}] \forall u \in D(\mathcal{A}^{m+1}) \cap H^s(\Omega).
\end{aligned}$$ □

Corollary 4.16. *Let $\mathcal{T}$ be a transient operator corresponding to different TG schemes, then the estimate of the total approximation error in energy norm is bounded by*

$$\|E\|_E \leq f(t^*)[\delta t^m \|\mathcal{A}^{m+1} u^0\|_E + h^{s-1} \|u^0\|_{H^s(\Omega)}] \tag{4.17}$$

for all $u^0 \in D(\mathcal{A}^{m+2}) \cap H^s(\Omega), s \geq 2$.

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