International Journal of Wavelets, Multiresolution
and Information Processing
Vol. 7, No. 2 (2009) 143–162

ERROR ESTIMATES FOR LINEAR PDEs SOLVED BY WAVELET BASED TAYLOR–GALERKIN SCHEMES

MANI MEHRA*

Department of Mathematics
Indian Institute of Technology
Delhi, 110016, India
mmehra@maths.iitd.ac.in

B. V. RATHISH KUMAR

Department of Mathematics
Indian Institute of Technology
Kanpur, U.P., 208016, India
bvrk@iitk.ac.in

Received 8 February 2008
Revised 28 December 2008

In this paper, we develop *a priori* and *a posteriori* error estimates for wavelet-Taylor–Galerkin schemes introduced in Refs. 6 and 7 (particularly wavelet Taylor–Galerkin scheme based on Crank–Nicolson time stepping). We proceed in two steps. In the first step, we construct the *priori* estimates for the fully discrete problem. In the second step, we construct error indicators for *posteriori* estimates with respect to both time and space approximations in order to use adaptive time steps and wavelet adaptivity. The space error indicator is computed using the equivalent norm expressed in terms of wavelet coefficients.

Keywords: Wavelets; time accurate schemes; time adaptivity; space adaptivity; error estimates.

AMS Subject Classification: 65M70, 65M17, 65T60

1. Introduction

The analysis of wavelet methods for partial differential equations is relatively recent development as compare to traditional methods like finite difference, finite element etc. The name wavelet or ondelette was coined some twenty five years ago by French researcher Grossmann and Morlet.[1] Since then, the growth of wavelet research in mathematics has been explosive with numerous contributing significantly due to its attractive features.

The key idea is that wavelet bases combine the advantages of both spectral and finite element methods. Moreover, wavelet elements capture localized phenomena

*Corresponding author.

and compute a multiscale solution to partial differential equations with higher convergence rates than conventional finite element methods. Adaptive techniques are also widely used for the solution of stationary problems, where phenomena requiring an adapted mesh may also appear. Kevlahan *et al.*[2–4] and Dahmen *et al.*[5] provide an overview of recent progress in the development and use of adaptive wavelet methods in fluid mechanics.

Numerically, the high order convergence rates and asymptotic analysis of a new class of wavelet based TG schemes have been established.[6,7] The fundamental concept behind the Taylor–Galerkin (TG) schemes (time accurate schemes) is to incorporate more analytic information in the numerical scheme in the most direct and natural way, so that the technique may be regarded as an extension to PDEs of the Obrechkoff methods for ODEs. In this paper, we are interested in derivation of *priori* and *posteriori* estimates for the discretization of parabolic equations, which relies on a wavelet Galerkin method with respect to space variables and a high order time accurate scheme with respect to time. The main conceptual ingredients for prior estimates using the spectral decomposition of self adjoint operators and wavelet approximation theory. The *priori* estimates for third order Taylor–Galerkin schemes (E-TGS) has been reported in Ref. 8 by authors.

For *posteriori* estimates we introduce time and space error indicators. The idea of space-time finite element adaptivity has been already discussed in Refs. 9 and 10. In the wavelet setting, one may decide whether to refine or not, depending on the size of the wavelet coefficients. In order to design a rigorous strategy, space error indicator is then used to decide in a more precise way which function we have to add or remove from the approximation space. The idea of space error indicator for elliptic problem has been discussed in Ref. 11. In this paper we follow an approach where the main idea consists of uncoupling of space and time errors for higher order time accurate schemes.

The paper is organized as follows. In Sec. 2, we summarize some basics of wavelet analysis. In Sec. 3, we consider the abstract formulation of the problem and discretization of problem. Sections 4 and 5 are devoted to *priori* error estimation and *posteriori* error analysis of the wavelet based time accurate schemes respectively.

2. General Setting

Definition 1. A multiresolution analysis (MRA) of $L_2(\mathbb{R}^\nu)$ describes a sequence of nested approximation spaces $V_j, j \in \mathbb{Z}$ such that closure of their union equals $L^2(\mathbb{R}^\nu)$. MRA is characterized by the following axioms

$$
\begin{gathered}
\{0\} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \cdots \subset L^2(\mathbb{R}^\nu) \\
\overline{\bigcup_{j=-\infty}^{j=\infty} V_j} = L^2(\mathbb{R}^\nu) \\
\bigcap_{j \in Z} V_j = 0 \\
f \in V_j \text{ if and only if } f(2(.)) \in V_{j+1} \\
\phi(x-k)_{k\in\mathbb{Z}} \text{ is an orthonormal basis for } V_0.
\end{gathered}
\tag{2.1}
$$

We define W_j to be the orthogonal complement of V_j in V_{j+1}, i.e. $V_j \perp W_j$ and

$$V_{j+1} = V_j + W_j. \tag{2.2}$$

There exists a function, which is called a scaling function $\phi(x) \in V_0$, such that the sequence $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)_{k\in\mathbb{Z}}$ is an orthonormal basis for V_j and $\psi_{j,k} = 2^{j/2}\psi(2^j x - k)_{k\in\mathbb{Z}}$ is an orthonormal basis for W_j. Mathematically, one introduces at each step j, the subspace W_j, defined as the orthogonal complement of V_j in V_{j+1}. Then one has the fundamental theorem proved by Mallat[13] and Meyer.[14]

Theorem 2.1. *There exists a function of* W_0 *such that* $\psi(x-k)$, $k \in \mathbb{Z}$ *is an orthonormal basis of* W_0. ψ *has the regularity properties*

$$\int x^k \psi(x)dx = 0 \quad \text{for } 0 \le k \le D/2 - 1. \tag{2.3}$$

As pointed out by Meyer[14] the MRA in $L^2(\mathbb{R})$ can be used in the periodic case $L^2([0,1])$ by introducing a standard periodization technique. This technique consists at each scale in folding, around the integer values, the wavelet $\psi_{j,k}$ and the scaling functions $\phi_{j,k}$ centered in $[0,1]$.

$\lambda = (j,k)$ an index in Z^2. By Λ we will indicate the set of all admissible indexes λ, and we can write any distribution $f \in H^{-1}(0,1)$ as $f = \mathcal{P}_{j_0} f + \sum_{\lambda\in\Lambda}\langle f, \psi_\lambda\rangle\psi_\lambda$ satisfying $j \ge j_0$. The approximation space $\mathcal{V}_\mathcal{N}$ will be constructed by choosing somehow a subset of indexes $\Lambda_\mathcal{N} \subset \Lambda$ and writing

$$\mathcal{V}_\mathcal{N} = V_{j_0} \cup \operatorname{span}\langle\psi_\lambda, \lambda \in \Lambda_\mathcal{N}\rangle.$$

Simple thresholding of the largest contribution in the wavelet composition provides a compressed solution f_h such that $f_h = \mathcal{P}_h f$, where $\mathcal{P}_h$ is the projection defined on the space $\mathcal{V}_\mathcal{N} = \sum_{\lambda\in\Lambda_\mathcal{N}} u_\lambda\psi_\lambda$ and $\Lambda_\mathcal{N} = \Lambda_\mathcal{N}(u, X)$ is the set of indices corresponding to the $\mathcal{N}$ largest contributions $\|u_\lambda\psi_\lambda\|_X$ for several interesting choices of X. We set $h = 2^{-j}$.

3. Higher-Order Taylor–Galerkin Schemes

Within the Hilbert space formulation the initial boundary value problem can be reinterpreted as an abstract Cauchy problem for a linear, self adjoint, positive definite, operator $\mathcal{A}$.

$$\begin{aligned} \frac{d}{dt}u + \mathcal{A}u = 0, &\quad 0 < t \le t^*, \\ u(0) = u^0, &\quad t = 0. \end{aligned} \tag{3.1}$$

For a variational formulation of this problem we introduce Sobolev spaces. Let $\Omega \subset \mathbb{R}^\nu$ be a bounded domain with periodic boundary $\Gamma = \partial\Omega$. We denote by $H = L^2(\Omega)$ the usual square integrable functions with inner product $(.,.)$ and by $H^s(\Omega), s \ge 0$, the corresponding Sobolev spaces.[15] We assume that $\mathcal{A} \in \mathcal{L}(V; V^*)$. By $(.,.)_{V^*\times V}$ we denote the extension of $(.,.)$ as duality pairing in $V^* \times V$, and by $\|.\|$, $\|.\|_V$, $\|.\|_{V^*}$ the norms in $L^2(\Omega)$, V, V^*, respectively, and $\|.\|_{\mathcal{A}}$ denote the graph norm $(\|.\|^2 + \|\mathcal{A}.\|^2)^{1/2}$. Then $D(\mathcal{A}) \subset V \subset V^*$. We associate with $\mathcal{A}$ the bilinear form $a(.,.) : V \times V \to \mathbb{C}$ via

$$a(u, v) = (\mathcal{A}u, v)_{V^* \times V}, \quad u, v \in V. \tag{3.2}$$

Then the form $a(.,.)$ is continuous,

$$|a(u, v)| \leq \alpha \|u\|_V \|v\|_V, \quad \forall u, v \in V \tag{3.3}$$

and we assume that it is coercive in the sense that

$$a(u, v) \geq \beta \|u\|_V^2 \tag{3.4}$$

for some $0 < \beta \leq \alpha < \infty$. Then $\mathcal{A} \in \mathcal{L}(V, V^*)$ is an isomorphism and $\|\mathcal{A}\|_{\mathcal{L}(V,V^*)} \leq \alpha$ The time derivative $\dot{u}(t)$ in (3.1) is understood in the weak sense; i.e. for $u \in L^2((0, t^*); V)$ we have $\dot{u} \in L^2((0, t^*); V^*)$ defined by

$$\int_J (\dot{u}(t), v)_{V^* \times V} \varphi(t) dt = -\int_J (u(t), v) \dot{\varphi}(t) dt \tag{3.5}$$

for every $v \in V$, $\varphi \in C_0^\infty((0, t^*))$. If the initial condition satisfies an additional regularity assumption $u^0 \in D(\mathcal{A})$ then the solution $u \in C^1((0, \infty); V^*) \cap C((0, \infty); V)$. We say then that u is a strict solution to the problem.

We record now some fundamental results concerning operator $\mathcal{A}$ and the existence and uniqueness of weak solution u:

(i) Operator $\mathcal{A}$ is self adjoint and, therefore, its spectrum lies on the real line and consists of a point spectrum and continuous spectrum. For a bounded domain, the spectrum of $\mathcal{A}$ consists of eigenvalues only. Except for the 0-eigenvalue, all the eigenvalues are of finite multiplicity and the corresponding eigenspaces $\{u_n\}$ are orthogonal.

(ii) Operator $\mathcal{A}$ admits a classical spectral decomposition

$$\mathcal{A}u = \int_{-\infty}^{\infty} \lambda dE_\lambda u \tag{3.6}$$

$$D(\mathcal{A}) = \left\{ u \in H : \int_{-\infty}^{\infty} \lambda^2 d\|E_\lambda u\|^2 < \infty \right\} \tag{3.7}$$

where E_λ is a uniquely defined spectral family of $\mathcal{A}$.[15]

(iii) A real solution u exists and is unique. Moreover, it is of the form

$$u(t) = e^{-\mathcal{A}t} u^0 = \int_{-\infty}^{\infty} e^{-\lambda t} dE_\lambda u^0. \tag{3.8}$$

In particular, it follows from (3.8) that the energy is conserved.

$$\|u(t)\|^2 = \int_{-\infty}^{\infty} |e^{-t\lambda}|^2 d(E_\lambda u^0, u^0) = \|u^0\|^2, \quad \forall t \geq 0. \tag{3.9}$$

3.1. *Approximation in time and space*

Low order time schemes do not allow a sufficiently accurate approximation of the exponential operator in (3.8), or, stated in other words, they do not properly account for the directional character of propagation of information in hyperbolic problems. Higher order time stepping schemes provide a better approximation to the exponential function in (3.8), and consequently allow a better account of the propagation of information along the characteristics. Such methods are based on third or higher in time thus the solution u must be sufficiently smooth. In order to obtain a second order method the Taylor series is taken as

$$\begin{aligned} \frac{u^n - u^{n-1}}{\delta t} &= u_t^{n-1} + \frac{\delta t}{2} u_{tt}^{n-1} + O(\delta t^2) \\ \frac{u^{n-1} - u^n}{\delta t} &= -u_t^n + \frac{\delta t}{2} u_{tt}^n + O(\delta t^2). \end{aligned} \tag{3.10}$$

Combining the above expressions gives

$$\frac{u^n - u^{n-1}}{\delta t} = \frac{1}{2}(u_t^{n-1} + u_t^n) + \frac{\delta t}{4}(u_{tt}^{n-1} - u_{tt}^n) + \cdots. \tag{3.11}$$

Which is a higher order wavelet-Taylor–Galerkin scheme based on Crank–Nicolson time stepping? First, using the original equations (3.1), we calculate the time derivatives in terms of spatial derivatives as follows

$$u_t = -\mathcal{A}u, \quad u_{tt} = \mathcal{A}^2 u. \tag{3.12}$$

Now putting these values in (3.11), the initial boundary value problem (3.1) is converted into a sequence of boundary value problems (3.13),

$$\begin{aligned} u^n &= \mathcal{T} u^{n-1} \\ u(x,0) &= u^0 \\ \delta t = t^*/N, \quad t_n &= n\delta t, \quad n = 1, \ldots, N+1. \end{aligned} \tag{3.13}$$

Consider $\mathcal{V}_{\mathcal{N}}^p \subset V$ of periodic version of space $\mathcal{V}_N$ and $\mathbb{V}_N^p = \mathcal{V}_{\mathcal{N}}^p \cap D(\mathcal{A})$. To discretize in space we use the wavelet projection $\mathcal{P}_h : V \to \mathbb{V}_N^p$. We consider the spatial approximation in the form

$$\begin{aligned} u_h^n &= \mathcal{T}_h u_h^{n-1} \\ u_h(x,0) &= u_h^0 \\ \delta t = t^*/N, \quad t_n &= n\delta t, \quad n = 1, \ldots, N+1. \end{aligned} \tag{3.14}$$

Now, multiplying by a test function $v_h \in \mathbb{V}_N^p$, integrating over Ω, and integrating by parts, we arrive at a variational formulation of the wavelet Taylor–Galerkin scheme based on Crank–Nicolson (C-TGS) time stepping:

Given $u_h^0 \in \mathbb{V}_N^p$

$$\mathcal{B}(u_h^n, v_h) - \frac{\delta t}{2}\mathcal{D}(u_h^n, v_h) + \frac{\delta t^2}{4}\mathcal{C}(u_h^n, v_h)$$
$$= \mathcal{B}(u_h^{n-1}, v_h) + \frac{\delta t}{2}\mathcal{D}(u_h^{n-1}, v_h) + \frac{\delta t^2}{4}\mathcal{C}(u_h^n, v_h) \tag{3.15}$$

where the bilinear forms $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are defined by

$$\begin{aligned} &\mathcal{B}, \mathcal{C}, \mathcal{D} : V \times V \to \mathbb{C} \\ &\quad \mathcal{B}(u, v) = (u, v), \quad \mathcal{C}(u, v) = (\mathcal{A}u, \mathcal{A}v), \quad \mathcal{D}(u, v) = (\mathcal{A}u, v) \end{aligned} \tag{3.16}$$

and $\mathcal{B}_1(.,.)$ is define by

$$\mathcal{B}_1(u, v) = \mathcal{B}(u^{n+1}, v) - \frac{\delta t}{2}\mathcal{D}(u^{n+1}, v) + \frac{\delta t^2}{4}\mathcal{C}(u^{n+1}, v).$$

Moreover, let us introduce the norm

$$[[u]](t) = \left(\|u(t)\|^2 + \int_0^t \left(- \left\| \frac{\delta t}{2} A^{\frac{1}{2}} u \right\|^2 + \left\| \frac{\delta t^2}{4} Au \right\|^2 \right) ds \right)^{\frac{1}{2}}. \tag{3.17}$$

It is easy to check that bilinear from $\mathcal{B}_1$ is continuous and coercive with respect to [[.]] norm. And therefore, by the virtue of the Lax–Milgram theorem bilinear form is well defined. For details of wavelet based TG schemes refer to Refs. 6 and 7.

4. *A Priori* Error Estimation for Wavelet Based Taylor–Galerkin Schemes

4.1. *Temporal approximation error*

Let $u(t)$ be the solution of the Cauchy problem (3.1) and let $u^n(t)$ be its semidiscrete approximation at time $t = n\delta t$ as in (3.13). To estimate the error in TG schemes, we need the following lemmas.

Lemma 4.1. *If $\mathcal{T}$ is a transient operator corresponding to different TG schemes, then $\mathcal{T}$ can be represented as a rational function of the underlying operator $\mathcal{A}$,*

$$\mathcal{T}u = r(\delta t \mathcal{A})u. \tag{4.1}$$

Proof. We prove the above assertion for C-TGS scheme. For a C-TGS scheme

$$\mathcal{T}u = \left(\mathcal{I} - \frac{\delta t}{2}\mathcal{A} + \frac{\delta t^2}{4}\mathcal{A}^2\right)^{-1} \left[I + \frac{\delta t}{2}\mathcal{A} + \frac{\delta t^2}{3}\mathcal{A}^2\right] u$$
$$= \int_{-\infty}^{\infty} r(\delta t \lambda) dE_\lambda u = r(\delta t \mathcal{A})u$$

where $r : \mathbb{Z} \to \mathbb{Z}$ is given by

$$r(z) = \left(1 - \frac{z}{2} + \frac{z^2}{4}\right)^{-1} \left(1 + \frac{z}{2} + \frac{z^2}{4}\right)$$
$$r(\mathbb{Z}) = P_1(z)/(1 - z/2 + z^2/4)$$

P_1 is a second order polynomial. Hence (4.1) follows. □

Lemma 4.2. *If $\mathcal{T}$ is a transient operator corresponding to different TG schemes, then for an mth-order TG scheme,*

$$|r(y) - e^{-y}| = O(|y^{m+1}|), \quad y \in \mathbb{R}. \tag{4.2}$$

Proof. Follows from the definition. □

Remark 4.3. We use the notion of asymptotic stability of a numerical method for a discrete problem of the form $du/dt = \mathcal{L}u$ where $\mathcal{L}$ is assumed to be a diagonal matrix. The region of absolute stability of a numerical method is defined for the scalar model problem $du/dt = \lambda u$ to be set of all $\lambda\delta t$ such that $\|u^n\|$ is bounded as $t \to \infty$. Finally we say that a numerical method is asymptotically stable for a particular problem if, for small $\delta t > 0$, the product of δt times every eigenvalues of $\mathcal{L}$ lies within the region of absolute stability. The region of absolute stability for wavelet-Taylor–Galerkin schemes are defined in Ref. 6.

Lemma 4.4. *If $\mathcal{T}$ is a transient operator corresponding to different TG schemes then, for asymptotically stable scheme there exists a δt such that*

$$|r(\delta t\lambda)| \leq 1. \tag{4.3}$$

We now define E_N, the temporal approximation error, $E_N = u(t^*) - u^N$. To estimate $\|E_N\|$ we need the following result.

Lemma 4.5. *Let $u \in D(\mathcal{A}^{m+1})$, $\mathcal{T}$ is a transient operator corresponding to different TG schemes and* (4.2) *holds. Then there exists a constant c such that the one step error satisfies*

$$\|e^{-\mathcal{A}\delta t}u - r(\delta t\mathcal{A})u\| \leq c\delta t^{m+1}\|\mathcal{A}^{m+1}u\|. \tag{4.4}$$

Proof.

$$\begin{aligned}\|e^{-\mathcal{A}\delta t}u - r(\delta t\mathcal{A})u\|^2 &= \int_{-\infty}^{\infty} |e^{-\lambda\delta t} - r(\lambda\delta t)|^2 d(E_\lambda u, u) \\ &\leq (c\delta t^{m+1})^2 \int_{-\infty}^{\infty} |\lambda^{m+1}|^2 d(E_\lambda u, u) \\ &= (c\delta t^{m+1})^2 \|\mathcal{A}^{m+1}u\|^2.\end{aligned}$$

□

The main consequence of the above lemmas can be formulated as follows.

Lemma 4.6. *Let $\mathcal{T}$ is a transient operator corresponding to different TG schemes and* (4.2) *and* (4.3) *holds then the error estimate is bounded by*

$$\|E_N\| \leq ct^*\delta t^m \|\mathcal{A}^{m+1}u^0\| \tag{4.5}$$

for all $u^0 \in D(\mathcal{A}^{m+1})$.

Proof.

$$\begin{aligned}\|E_N\| &= \|e^{-N\mathcal{A}\delta t}u^0 - r^N(\delta t\mathcal{A})u^0\| \\ &\le \sum_{j=0}^{N-1} \|e^{-(N-j-1)\delta t\mathcal{A}}\|\|r^j(\delta t\mathcal{A})\|\|e^{-\mathcal{A}\delta t}u^0 - r(\delta t\mathcal{A})u^0\| \\ &\le N\|e^{-\mathcal{A}\delta t}u^0 - r(\delta t\mathcal{A})u^0\| \quad \text{(using Lemma (4.5))} \\ &\le Nc\delta t^{m+1}\|\mathcal{A}^{m+1}u^0\| \\ &= ct^*\delta t^m\|\mathcal{A}^{m+1}u^0\|, \quad u^0 \in D(\mathcal{A}^{m+1})\end{aligned}$$

which completes the proof. □

We also estimate temporal approximation error in the energy norm $\|.\|_E$ defined by the bilinear form $\|.\|_E = \mathcal{B}_1(.,.)^{1/2}$ where for C-TGS scheme $\mathcal{B}_1(.,.) = \mathcal{B}(.,.) - \frac{\delta t}{2}\mathcal{D} + \frac{\delta t^2}{4}\mathcal{C}(.,.)$.

Corollary 4.7. *Let $\mathcal{T}$ is a transient operator corresponding to different TGS schemes and* (4.2) *and* (4.3) *holds then,*

$$\|E_N\|_E \le ct^*\delta t^m\|\mathcal{A}^{m+1}u^0\|_E \tag{4.6}$$

for all $u^0 \in D(\mathcal{A}^{m+2})$.

Proof. Assume $u \in D(\mathcal{A}^{m+2})$ then one step error is estimated by

$$\begin{aligned}\|e^{-\mathcal{A}\delta t}u - r(\delta t\mathcal{A})u\|_E^2 &= \int_{-\infty}^{\infty} |e^{-\lambda\delta} - r(\lambda\delta t)|^2(1 - \delta t/2 + \delta t^2/4)d(E_\lambda u, u) \\ &\le (c\delta t^{m+1})^2 \int_{-\infty}^{\infty} |\lambda^{m+1}|^2(1 - \delta t/2 + \delta t^2/4)d(E_\lambda u, u) \\ &= (c\delta t^{m+1})^2\|\mathcal{A}^{m+1}u\|_E^2.\end{aligned}$$

Next, we note that

$$\|e^{-\delta t\mathcal{A}}\|_E^2 = \|u\|_E^2, \quad \|r(\delta t\mathcal{A})u\|_E^2 \le \|u\|_E^2$$

Consequently, assuming $u^0 \in D(\mathcal{A}^{m+2})$, estimate (4.5) is also valid in the energy norm. □

4.2. *Spatial approximation error*

We now define $E_h = u^N - u_h^N$, the spatial approximation error. The rate of convergence of a wavelet scheme to the exact solution with respect to the number N of degrees of freedom i.e. the number of wavelets which are used to describe the solution is the same as the rate of convergence of the best N-term approximation which would be obtained by retaining the N largest wavelet coefficients of the exact solution.

Definition 2. An MRA of $L^2(\mathbb{R}^\nu)$ is said to be r-regular ($r \in \mathbb{N}$) if the function ϕ is r-regular, that is, for each $m \in \mathbb{N}$ there exists c_m such that for all multi-indexes α, $|\alpha| \leq r$, the following conditions holds: $|D^\alpha \phi(x)| \leq c_m(1+|x|)^{-m}$.

Now we have the following wavelet approximation theorem.[18]

Theorem 4.8. *Let $r-1 \leq q \leq r$, $-r \leq s \leq r+1$ and $q \leq s$. Then*

$$\|u - \mathcal{P}_h u\|_{H^q(\Omega)} \leq \tilde{c} h^{s-q} \|u\|_{H^s(\Omega)} \tag{4.7}$$

for all $u \in H^s(\Omega)$, where $\tilde{c}$ is a constant independent of h and u.

This lemma shows that a smoothness property implies good approximation property of wavelet projection operator. Now we first record the following inequality using Theorem 4.8.

Lemma 4.9. *Let $\mathcal{T}$ is a transient operator corresponding to different TG schemes, then*

$$\|\mathcal{P}_h \mathcal{T} u - \mathcal{T}_h \mathcal{P}_h u\| \leq c^* h^s \|u\|_{H^s(\Omega)} \tag{4.8}$$

for all $u \in H^s(\Omega)$, where c^* is a constant independent of h and u.

Proof.

$$\begin{aligned}
\|\mathcal{P}_h \mathcal{T} u - \mathcal{T}_h \mathcal{P}_h u\| &= \|\mathcal{P}_h \mathcal{T} u - \mathcal{P}_h \mathcal{T} \mathcal{P}_h u\| \\
&\leq \|\mathcal{P}_h\| \|\mathcal{T} u - \mathcal{T} \mathcal{P}_h u\| \\
&\leq \|\mathcal{T} u - \mathcal{T} \mathcal{P}_h u\| \\
&\leq \|\mathcal{T}\| \|u - \mathcal{P}_h u\| \\
&\leq \|\mathcal{T}\| \|u - \mathcal{P}_h u\| \\
&\leq M \|u - \mathcal{P}_h u\| \\
&\leq c^* h^s \|u\|_{H^s(\Omega)}.
\end{aligned}$$

□

We note the uniform quasi-boundedness of $\mathcal{T}_h^N$ by,

$$\|\mathcal{T}_h^N\| = \sup_{u \in V_{Pj}, u \neq 0} \frac{\|T_h^N u\|}{\|u\|} \leq L e^{Bt^*} \text{ independent of } N.$$

Now, using the above lemmas spatial approximation error $\|E_h\|$ can be bounded as follows

Lemma 4.10. *Let $\mathcal{T}$ is a transient operator corresponding to different TG schemes, then*

$$\|E_h\| \leq N L e^{Bt^*} c^* h^s \|\mathcal{T}^{I-1} u^0\|_{H^s(\Omega)} + c^* h^s \|\mathcal{T}^N u^0\|_{H^s(\Omega)} \tag{4.9}$$

for all $u^0 \in H^s(\Omega)$.

Proof.

$$\begin{aligned}
\|E_h\| &= \|\mathcal{T}^N u^0 - \mathcal{T}_h^N \mathcal{P}_h u^0\| \\
&\leq \sum_{i=1}^{i=N} \|\mathcal{T}_h^{N-1}\| \|(\mathcal{P}_h \mathcal{T} - \mathcal{T}_h \mathcal{P}_h)\mathcal{T}^{i-1} u^0\| + \|\mathcal{T}^N u^0 - \mathcal{P}_h \mathcal{T}^N u^0\| \\
&\leq NLe^{Bt^*} \|(\mathcal{P}_h \mathcal{T} - \mathcal{T}_h \mathcal{P}_h)\mathcal{T}^{I-1} u^0\| + \|\mathcal{T}^N u^0 - \mathcal{P}_h T^N u^0\| \\
&\leq NLe^{Bt^*} c^* h^s \|\mathcal{T}^{I-1} u^0\|_{H^s(\Omega)} + c^* h^s \|\mathcal{T}^N u^0\|_{H^s(\Omega)}, \quad \forall u^0 \in H^s(\Omega)
\end{aligned} \tag{4.10}$$

where

$$\|(\mathcal{P}_h \mathcal{T} - \mathcal{T}_h \mathcal{P}_h)\mathcal{T}^{I-1} u^0\| = \max_{i=1,2,\ldots,N} \|\mathcal{P}_h \mathcal{T} - \mathcal{T}_h \mathcal{P}_h)\mathcal{T}^{i-1} u^0\| \qquad \square$$

Now we will prove the same wavelet error estimate for the energy norm in the subsequent lemmas. First, we prove the approximation error of the wavelet projection in the energy norm as follows

Lemma 4.11. *Let $r-1 \leq q \leq r$, $-r \leq s \leq r+1$ and $s \geq 1$, $q \leq s$. Then*

$$\|u - \mathcal{P}_h u\|_E \leq \check{c} h^s \|u\|_{H^s(\Omega)}$$

for all $u \in D(\mathcal{A}) \cap H^s(\Omega)$, where $\check{c}$ is a constant independent of h and u.

Proof.

$$\|u - \mathcal{P}_h\|_E = \inf_{\chi \in V_h} \{\|u - \chi\|_E\} \quad \text{for C-TGS scheme}$$

$$\begin{aligned}
\|u - \mathcal{P}_h\|_E = \inf_{\chi \in V_h} &\{(\|u - \chi\|^2 - (\delta t/2)\|A^{\frac{1}{2}}(u - \chi)\| \\
&+ (\delta t^2/4)\|\mathcal{A}(u - \chi)\|^2)^{1/2}\} \\
\leq \inf_{\chi \in V_h} &\{\|u - \chi\| - (\sqrt{(\delta t)}/\sqrt{(2)})\|A^{\frac{1}{2}}(u - \chi)\| \\
&+ (\delta t/\sqrt{(4)})\|A(u - \chi)\|\} \\
\leq \check{c} h^s &\|u\|_{H^s(\Omega)} \quad \forall u \in D(\mathcal{A}) \cap H^s(\Omega) \text{ using (4.7).}
\end{aligned} \tag{4.11}$$

$\square$

Now we first record the following inequality using lemma 4.11.

Lemma 4.12. *Let $\mathcal{T}$ is a transient operator corresponding to different TG schemes, then*

$$\|\mathcal{P}_h \mathcal{T} u - \mathcal{T}_h \mathcal{P}_h u\|_E \leq \check{c} h^s \|u\|_{H^s(\Omega)} \tag{4.12}$$

for all $u \in D(\mathcal{A}) \cap H^s(\Omega)$, $s \geq 1$ where $\check{c}$ is a constant independent of h and u.

Proof.

$$\begin{aligned}\|\mathcal{P}_h\mathcal{T}u - \mathcal{T}_h\mathcal{P}_h u\|_E &= \|\mathcal{P}_h\mathcal{T}u - \mathcal{P}_h\mathcal{T}\mathcal{P}_h u\|_E \\ &\le \|\mathcal{P}_h\|_E\|\mathcal{T}u - \mathcal{T}\mathcal{P}_h u\|_E \\ &\le \|\mathcal{T}u - \mathcal{T}\mathcal{P}_h u\|_E \\ &\le \|\mathcal{T}\|_E\|u - \mathcal{P}_h u\|_E \\ &\le \|\mathcal{T}\|_E\|u - \mathcal{P}_h u\|_E \\ &\le M\|u - \mathcal{P}_h u\|_E \\ &\le \check{c}h^s\|u\|_{H^s(\Omega)}.\end{aligned}$$

□

We note that if the uniform quasi-boundedness of $\mathcal{T}_h^N$ is also true in the energy norm, then the spatial approximation error $\|E_h\|_E$ is bounded as follows.

Lemma 4.13. *Let $\mathcal{T}$ is a transient operator corresponding to different TG schemes, then*

$$\|E_h\|_E \le NLe^{Bt^*}\check{c}h^s\|\mathcal{T}^{I-1}u^0\|_{H^s(\Omega)} + \check{c}h^s\|\mathcal{T}^N u^0\|_{H^s(\Omega)} \tag{4.13}$$

for all $u^0 \in D(\mathcal{A}) \cap H^s(\Omega)$, $s \ge 2$.

Proof.

$$\begin{aligned}\|E_h\|_E &= \|\mathcal{T}^N u^0 - \mathcal{T}_h^N\mathcal{P}_h u^0\|_E \\ &\le \sum_{i=1}^{i=N}\|\mathcal{T}_h^{N-i}\|_E\|(\mathcal{P}_h\mathcal{T} - \mathcal{T}_h\mathcal{P}_h)\mathcal{T}^{i-1}u^0\|_E + \|\mathcal{T}^N u^0 - \mathcal{P}_h\mathcal{T}^N u^0\|_E \\ &\le LNe^{Bt^*}\|\mathcal{P}_h\mathcal{T} - \mathcal{T}_h\mathcal{P}_h)\mathcal{T}^{I-1}u^0\|_E + \|\mathcal{T}^N u^0 - \mathcal{P}_h T^N u^0\|_E \\ &\le LNe^{Bt^*}\check{c}h^s\|\mathcal{T}^{I-1}u^0\|_{H^s(\Omega)} + \check{c}h^s\|\mathcal{T}^N u^0\|_{H^s(\Omega)}.\end{aligned} \tag{4.14}$$

□

We are now in a position to prove the main theorem of this paper for *priori* estimates.

Theorem 4.14. *Let $\mathcal{T}$ is a transient operator corresponding to different TG schemes, then the estimate of the total approximation error is bounded by*

$$\|E\| \le f(t^*)[\delta t^m\|\mathcal{A}^{m+1}u^0\| + h^{s-1}\|u^0\|_{H^s(\Omega)}] \tag{4.15}$$

for all $u^0 \in D(\mathcal{A}^{m+1}) \cap H^s(\Omega)$.

Proof.

$$\begin{aligned}\|E\| &= \|u(t^*) - u_{\tau h}(t^*)\| \\ &\le \|E_\tau\| + \|E_h\| \\ &\le ct^*\delta t^m\|\mathcal{A}^{m+1}u^0\| + NLe^{bt^*}\check{c}h^s\|\mathcal{T}^{I-1}u^0\|_{H^s(\Omega)} + \check{c}h^s\|\mathcal{T}^N u^0\|_{H^s(\Omega)} \\ &\le f(t^*)[\delta t^m\|\mathcal{A}^{m+1}u^0\| + h^{s-1}\|u\|_{H^s(\Omega)}] \quad \forall u \in D(\mathcal{A}^{m+1}) \cap H^s(\Omega).\end{aligned}$$

□

Corollary 4.15. *Let $\mathcal{T}$ is a transient operator corresponding to TG schemes, then the estimate of the total approximation error in energy norm is bounded by*

$$\|E\|_E \leq f(t^*)[\delta t^m \|\mathcal{A}^{m+1}u^0\|_E + h^{s-1}\|u^0\|_{H^s(\Omega)}] \tag{4.16}$$

for all $u^0 \in D(\mathcal{A}^{m+2}) \cap H^s(\Omega), s \geq 2$.

5. *A Posteriori* Estimation for Wavelet Taylor–Galerkin Schemes

Although the approach presented below is quite general, we restrict our attention to a simple heat diffusion problem which is a special case of (3.1) where $\mathcal{A} = -\Delta$.

5.1. *The time semi-discrete problem*

In order to describe the (possibly adaptive) time discretization of Eq. (3.1), we introduce a partition of the interval $[0, T]$ into subintervals $[t_{n-1}, t_n]$, $1 \leq n \leq N$, such that $0 = t_0 < t_1 < \cdots < t_N = T$. We denote by τ_n the length $t_n - t_{n-1}$, by τ the N-tuple $(\tau_1, \ldots, \tau_N)$ and by $|\tau|$ the maximum of the τ_n, $1 \leq n \leq N$. When the mesh is uniform we can say $\tau_n = \delta t$. We also define the regularity parameter

$$\sigma_\tau = \max_{2 \leq n \leq N} \frac{\tau_n}{\tau_{n-1}}. \tag{5.1}$$

With each family $(v^n)_{0 \leq n \leq N}$, we agree to associate the function v_{τ_1} on $[0, T]$ which is affine on each interval $[t_{n-1}, t_n]$, $1 \leq n \leq N$, and equal to v^n at t_n, $0 \leq n \leq N$, as

$$\forall t \in [t_{n-1}, t_n], \quad v_\tau(t) = v^n - \frac{t_n - t}{\tau_n}(v^n - v^{n-1}). \tag{5.2}$$

The semi–discrete problem derived from the C-TGS scheme is now written as

$$\begin{aligned} (u^n, v) - \frac{\tau_n}{2}(\nabla u^n, \nabla v) + \frac{\tau_n^2}{4}(\Delta u^n, \Delta v) \\ = (u^{n-1}, v) + \frac{\tau_n}{2}(\nabla u^{n-1}, \nabla v) + \frac{\tau_n^2}{4}(\Delta u^{n-1}, \Delta v). \end{aligned} \tag{5.3}$$

By taking v equal to u^n in (5.3), we get

$$\|u^n\|^2 - \frac{\tau_n}{2}\|\nabla u^n\|^2 + \frac{\tau_n^2}{4}\|\Delta u^n\| \leq \|u^{n-1}\|^2 + \frac{\tau_n}{2}\|\nabla u^{n-1}\|^2 + \frac{\tau_n^2}{4}\|\Delta u^{n-1}\|. \tag{5.4}$$

We now define the norm on whole sequences $(v^m)_{1 \leq m \leq n}$ by

$$[[(v^m]]_n \leq \left(\|u_0\|^2 + \sum_{m=1}^{n} -\frac{\tau_m}{2}\|\nabla u^m\|^2 + \frac{\tau_m}{4}\|\Delta u^m\|\right)^{\frac{1}{2}}. \tag{5.5}$$

The equivalence of two norms $[[u^m]]_n$ and $[[u_\tau]](t_n)$ as introduced in (3.17) is proved in the next lemma.

Lemma 5.1. *If $(v^n)_{0\le n\le N}$ is in $H^2(\Omega)^{N+1}$, then the following equivalence property holds for $|\tau| < \alpha_0$, where α_0 is a positive real number*

$$\frac{1}{4}[[v^m]]_n^2 \le [[v_\tau]]^2(t_n) \le \frac{1}{2}(1+\sigma_\tau)[[v^m]]_n^2 + \frac{1}{4}\tau_1\|\nabla v^0\|^2 + \frac{\tau_m}{8}\tau_1\|\Delta v^0\|^2. \qquad (5.6)$$

Proof. Owing to definitions (5.5) and (3.17), we have to compare the quantities

$$X_m = \int_{t_{m-1}}^{t_m}\left(-\frac{1}{2}\|\nabla v_\tau(s)\|^2 + \frac{\tau_m}{4}\|\Delta v_\tau(s)\|^2\right)ds \quad \text{and}$$

$$Y_m = -\frac{\tau_m}{2}\|\nabla v^m\|^2 + \frac{\tau_m^2}{4}\|\Delta v^m\|^2.$$

It can also be noted that, for a.e x in Ω

$$\int_{t_{m-1}}^{t_m}|\nabla v_\tau(x,s)|^2 ds = \frac{\tau_m}{3}(|\nabla v^m(x)|^2 + |\nabla v^{m-1}(x)|^2 + \nabla v^m.\nabla v^{m-1}) \qquad (5.7)$$

and

$$\int_{t_{m-1}}^{t_m}|\Delta v_\tau(x,s)|^2 ds = \frac{\tau_m}{3}(|\Delta v^m(x)|^2 + |\Delta v^{m-1}(x)|^2 + \Delta v^m.\Delta v^{m-1}). \qquad (5.8)$$

Therefore

$$\begin{aligned} X_m = &-\frac{\tau_m}{6}(\|\nabla v^m(x)\|^2 + \|\nabla v^{m-1}(x)\|^2 + \nabla v^m.\nabla v^{m-1}) \\ &+ \frac{\tau_m^2}{12}(\|\Delta v^m(x)\|^2 + \|\Delta v^{m-1}(x)\|^2 + \Delta v^m.\Delta v^{m-1}). \end{aligned} \qquad (5.9)$$

So using the inequality $ab \ge -\frac{a^2}{4} - b^2$ yields

$$X_m \ge \frac{\tau_m}{4}\left(-\frac{1}{2}\|\nabla v^m\|^2 + \frac{\tau_m}{4}\|\Delta v^m\|\right) = \frac{Y_m}{4}$$

whence the first inequality in (5.6). Similarly, by using the inequality $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$, we have

$$X_m \le \frac{\tau_m}{2}\left(-\frac{1}{2}(\|\nabla v^m\|^2 + \|\nabla v^{m-1}\|^2) + \frac{\tau_m}{4}(\|\Delta v^m\|^2 + \|\Delta v^{m-1}\|^2)\right).$$

When $m = 1$, we keep it as such. When $m > 1$, we use the regularity parameter as introduced in (5.1) to obtain

$$\begin{aligned} X_m \le &\frac{\tau_m}{2}\left(-\frac{1}{2}\|\nabla v^m\|^2 + \frac{\tau_m}{4}\|\Delta v^m\|^2\right) \\ &+ \frac{\tau_{m-1}\sigma_\tau}{2}\left(-\frac{1}{2}\|\nabla v^{m-1}\|^2 + \frac{\tau_{m-1}\sigma_\tau}{4}\|\Delta v^{m-1}\|^2\right). \end{aligned}$$

By summing up the previous lines on m, we derive the second inequality in (5.6). □

We are now interested in finding a time error indicator and studying its equivalence with the error. For each n, $1 \le n \le N$, we define the time error indicator

$$\eta_n = \frac{\tau_n}{3}\left(-\frac{1}{2}\|\nabla(u_h^n - u_h^{n-1})\| + \frac{\tau_n}{4}\|\Delta(u_h^n - u_h^{n-1})\|\right). \tag{5.10}$$

Now we estimate $[[u - u_\tau]](t_n)$ by the following lemma.

Lemma 5.2. *Assume there exists a positive constant α_1, such that, when both $|\tau|$ and h are smaller than α_1, then the following a posteriori error estimate holds between the solution u of the problem* (3.1) *and the solution $(u^n)_{0\le n\le N}$ of problem* (5.3), *for all t_n, $1 \le n \le N$*

$$[[u - u_\tau]](t_n) \le c([[u_\tau - u_{h\tau}]](t_n) + \left(\sum_{m=1}^{n} \eta_m^2\right)^{\frac{1}{2}}. \tag{5.11}$$

Proof. When applying Eq. (3.1) to the function u_τ, we obtain for all t in $[t_{n-1}, t_n]$ and $v \in H_0^2(\Omega)$

$$\begin{aligned}(\partial_t u_\tau(t), v) - (\nabla u_\tau, \nabla v) = {} & \left(\frac{u^n - u^{n-1}}{\tau_n}, v\right) - \frac{1}{2}\left(\nabla u^n + \nabla u^{n-1}, \nabla v\right) \\ & + \frac{\tau_n}{4}(\Delta u^n - \Delta u^{n-1}, \Delta v) - \frac{1}{2}(\nabla u_\tau - \nabla u^n \\ & + \nabla u_\tau - \nabla u^{n-1}, \nabla v) + \frac{\tau_n}{4}(\Delta u_\tau - \Delta u^n - \Delta u_\tau \\ & + \Delta u^{n-1}, \Delta v)\end{aligned} \tag{5.12}$$

$$\begin{aligned}(\partial_t u_\tau(t), v) - (\nabla u_\tau, \nabla v) = {} & -\frac{1}{2}(\nabla u_\tau - \nabla u^n + \nabla u_\tau - \nabla u^{n-1}, \nabla v) \\ & + \frac{\tau_n}{4}(\Delta u_\tau - \Delta u^n - \Delta u_\tau + \Delta u^{n-1}, \Delta v).\end{aligned} \tag{5.13}$$

Thus, subtracting this line from original equation

$$\begin{aligned}(\partial_t(u - u_\tau)(t), v) - (\nabla(u - u_\tau), \nabla v) = {} & -\frac{1}{2}(\nabla u_\tau - \nabla u^n + \nabla u_\tau - \nabla u^{n-1}, \nabla v) \\ & + \frac{\tau_n}{4}(\Delta u_\tau - \Delta u^n - \Delta u_\tau + \Delta u^{n-1}, \Delta v).\end{aligned} \tag{5.14}$$

We now take v equal to $(u - u_\tau(t))$, integrate this line on $[t_{n-1}, t_n]$ and sum up on the n. By noting that $u - u_\tau$ vanishes at $t = 0$, this yields

$$\begin{aligned}\frac{1}{2}[[u - u_\tau(t)]](t_n) = \sum_{m=1}^{n} - & \left(\int_{t_{m-1}}^{t_m} \frac{1}{2}(\nabla u_\tau - \nabla u^m, \nabla v(s) ds\right) \\ & + \left(\int_{t_{m-1}}^{t_m} \frac{1}{2}(\nabla u^{m-1} - \nabla u_\tau, \nabla v(s)) ds\right)\end{aligned}$$

$$- \left(\int_{t_{m-1}}^{t_m} \frac{\tau_m}{4} (\Delta u^m - \Delta u_\tau, \Delta v(s)) ds \right)$$
$$+ \left(\int_{t_{m-1}}^{t_m} \frac{\tau_m}{4} (\Delta u^{m-1} - \Delta u_\tau, \Delta v(s)) ds \right). \tag{5.15}$$

Now we evaluate each term separately

$$\left| \int_{t_{m-1}}^{t_m} \frac{1}{2} (\nabla u_\tau - \nabla u^m, \nabla v(s)) ds \right|$$
$$\leq \frac{1}{2} \left(\int_{t_{m-1}}^{t_m} \|(\nabla u_\tau - \nabla u^m\|^2 ds \right)^{\frac{1}{2}} \left(\int_{t_{m-1}}^{t_m} \|\nabla v(s) ds)\|^2 ds \right)^{\frac{1}{2}}. \tag{5.16}$$

Note that

$$\left(\sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} \|v(s)\|^2 ds)^{\frac{1}{2}} \leq [[u - u_\tau]](t_n) \right) \tag{5.17}$$

and by definition of u_τ, we have

$$\left(\int_{t_{m-1}}^{t_m} \|(\nabla u_\tau - \nabla u^m)\|^2 ds \right)^{\frac{1}{2}} = \left(\frac{\tau_m}{3} \right)^{\frac{1}{2}} \|\nabla (u^m - u^{m-1})\|. \tag{5.18}$$

Then adding and subtracting ∇u_h^m and u_h^{m-1} in the right-hand side of Eq. (5.18), we get

$$\left(\int_{t_{m-1}}^{t_m} \|(\nabla u_\tau - \nabla u^m)\|^2 ds \right)^{\frac{1}{2}} \leq \left(\frac{\tau_m}{3} \right)^{\frac{1}{2}} \|\nabla (u_h^m - u_h^{m-1})\|$$
$$+ \left(\frac{\tau_m}{3} \right)^{\frac{1}{2}} \|\nabla (u^m - u_h^m)\|$$
$$+ \left(\frac{\tau_m}{3} \right)^{\frac{1}{2}} \|\nabla (u^{m-1} - u_h^{m-1})\|. \tag{5.19}$$

Similarly we can prove for the second term

$$\left(\int_{t_{m-1}}^{t_m} \|(\Delta u_\tau - \Delta u^m)\|^2 ds \right)^{\frac{1}{2}} \leq \left(\frac{\tau_m}{3} \right)^{\frac{1}{2}} \|\Delta \left(u_h^m - u_h^{m-1} \right)\|$$
$$+ \left(\frac{\tau_m}{3} \right)^{\frac{1}{2}} \|\Delta (u^m - u_h^m)\|$$
$$+ \left(\frac{\tau_m}{3} \right)^{\frac{1}{2}} \|\Delta (u^{m-1} - u_h^{m-1})\| \tag{5.20}$$

by adding (5.19) and (5.20) we get

$$-\frac{1}{2}\left(\int_{t_{m-1}}^{t_m}\|(\nabla u_\tau-\nabla u^m)\|^2 ds\right)^{\frac{1}{2}}+\frac{\tau_m}{4}\left(\int_{t_{m-1}}^{t_m}\|(\Delta u_\tau-\Delta u^m)\|^2 ds\right)^{-\frac{1}{2}}$$
$$\le\left(\frac{\tau_m}{3}\right)^{\frac{1}{2}}\left(-\frac{1}{2}\|\nabla(u_h^m-u_h^{m-1})\|+\frac{\tau_m}{4}\|\Delta(u_h^m-u_h^{m-1})\|\right)$$
$$+\left(\frac{\tau_m}{3}\right)^{\frac{1}{2}}\left(-\frac{1}{2}\|\nabla(u^m-u_h^m)\|+\frac{\tau_m}{4}\|\Delta(u^m-u_h^m)\|\right)$$
$$+\left(\frac{\tau_m}{3}\right)^{\frac{1}{2}}\left(-\frac{1}{2}\|\nabla(u^{m-1}-u_h^{m-1})+\frac{\tau_m}{4}\|\Delta(u^{m-1}-u_h^{m-1})\|\right) \tag{5.21}$$

using (5.6) we see that sum over m of the square of last four terms can be bounded by $2[[u_\tau-u_{\tau h}]]^2(t_n)$. Combining all this yields the desired results. Now we will prove the upper bound for the time indicator by the following theorem. □

Theorem 5.3. *Assume that the function* $u_0\in H_0^2(\Omega)$. *Then the following estimates holds for the indicator* η_m *defined in* (5.10), $1\le m\le N$

$$\eta_m\le([[u^m-u_h^m]]+\sigma_\tau^{\frac{1}{2}}[[u^m-u_h^{m-1})]]+c(\|\nabla(u-u_\tau)\|_{L^2(t_{n-1},t_n;L^2(\Omega))}$$
$$+\|\partial_t(u-u_\tau)\|_{L^2(t_{n-1},t_n;H^{-1}(\Omega))}). \tag{5.22}$$

Proof. We use the triangle inequality

$$\eta_m\le-\frac{1}{2}\left(\frac{\tau_m}{3}\right)^{\frac{1}{2}}\left(\|\nabla(u^m-u^{m-1})\|+\|\nabla(u^m-u_h^m)\|+\|\nabla(u^{m-1}-u_h^{m-1})\|\right)$$
$$\frac{\tau_m}{4}\left(\frac{\tau_m}{3}\right)^{\frac{1}{2}}\left(\|\Delta(u^m-u^{m-1})\|+\|\Delta(u^m-u_h^m)\|+\|\Delta(u^{m-1}-u_h^{m-1})\|\right). \tag{5.23}$$

Hence

$$\eta_m\le-\frac{1}{2}\left(\frac{\tau_m}{3}\right)^{\frac{1}{2}}\|\nabla(u^m-u^{m-1})\|+\frac{\tau_m}{4}\left(\frac{\tau_m}{3}\right)^{\frac{1}{2}}\|\Delta(u^m-u^{m-1})\|$$
$$+[[u^m-u_h^m]]+\sigma_\tau^{\frac{1}{2}}[[u^m-u_h^{m-1})]]. \tag{5.24}$$

Now to evaluate first term we take v in (5.14) equal to u^n-u^{n-1} and integrate between t_{n-1} and t_n. By noting that

$$-\frac{1}{2}\int_{t_{n-1}}^{t_n}(\nabla(u_\tau(s)-\nabla u^n),\nabla v)ds+\frac{\tau_m}{4}\int_{t_{n-1}}^{t_n}(\Delta(u_\tau(s)-\Delta u^{n-1}),\Delta v)ds$$
$$=\tau_n\left(-\frac{1}{2}\|\nabla(u^n-u^{n-1})\|^2+\frac{\tau_n}{4}\|\Delta(u^n-u^{n-1})\|^2\right), \tag{5.25}$$

$$\frac{\tau_n}{2}\|\nabla(u^n-u^{n-1})\|^2+\frac{\tau_m}{4}\|\Delta(u^n-u^{n-1})\|^2$$
$$=(\partial_t(u-u_\tau(t)),v)-(\nabla(u-u_\tau),\nabla v). \tag{5.26}$$

In Eq. (5.26) last two terms are bounded by $\|\partial_t(u-u_\tau)\|_{L^2(t_{n-1},t_n;H^{-1}(\Omega))}$ and $\|\nabla(u-u_\tau)\|_{L^2(t_{n-1},t_n;L^2(\Omega))}$ respectively. Combining all this yields the desired results. □

5.2. *The time and space semi–discrete problem*

For the space error indicator first we state the following theorem[19]

Theorem 5.4. *Let $f \in H^{-1}(\Omega)$ and let $-r < s < r$ be an integer verifying $|s| < 1$. Then*

(a) *If $s \geq 1$, then $f \in H_0^s$ iff $\|Q_j f\|_0 = \epsilon_j$ with $\epsilon_j \in l^2$.*
(b) *If $s \leq 0$, then $f \in H^s(0,1)$ iff $2^{js}\|Q_j f\|_0 = \epsilon_j$ with $\epsilon_j \in l^2$. In both cases, we have the norm equivalence*

$$\|f\|_s \approx \left(2^{2j_0}\|P_j f\|_0^2 + \sum_{j\geq j_0} 2^{2js}\|Q_j f\|_0^2\right)^{\frac{1}{2}}. \tag{5.27}$$

Find $v_h \in V_h$ verifying

$$\begin{aligned}(u_h^n, v_h) &- \frac{\tau_n}{2}(\nabla u_h^n, \nabla v_h) + \frac{\tau_n^2}{4}(\Delta u_h^n, \Delta v_h)\\ &= (u_h^{n-1}, v_h) + \frac{\tau_n}{2}(\nabla u_h^{n-1}, \nabla v_h) + \frac{\tau_n^2}{4}(\Delta u_h^{n-1}, \Delta v_h).\end{aligned} \tag{5.28}$$

We introduce the residual

$$r_h = u_h^{n-1} + \frac{\tau_n}{2}\Delta u_h^{n-1} + \frac{\tau_n^2}{4}\Delta^2 u_h^{n-1} - u_h^n + \frac{\tau_n}{2}\Delta u_h^n - \frac{\tau_n^2}{4}\Delta^2 u_h^n \in H^{-1}(0,1).$$

Then we can bound

$$\|u^{n+1} - u_h^{n+1}\| + \frac{\tau_n}{2}\|u^{n+1} - u_h^{n+1}\|_1 + \frac{\tau_n^2}{4}\|u^{n+1} - u_h^{n+1}\|_2 \leq C\|r_h^n\|_{-1}. \tag{5.29}$$

We need then to estimate the H^{-1} norm of residual using Theorem 5.4.

$$\|r_h^n\|_{-1}^2 \leq C\sum_{\lambda\in\wedge} 2^{-2j}|\langle r_h^n, \psi_\lambda\rangle|^2 = C\sum_{\lambda\in\wedge\setminus\wedge_h} 2^{-2j}|\langle r_h^n, \psi_\lambda\rangle|^2. \tag{5.30}$$

Since $u_h^n = P_{j_0}u^n + \sum_{\lambda\in\wedge_h} u_\lambda^n \psi_{\lambda'}$ we have

$$\begin{aligned}|\langle r_h^n, \psi_\lambda\rangle| &= |\langle u_h^n + \Delta u_h^n + \Delta^2 u_h^n, \psi_\lambda\rangle|\\ &= \left|u_{\lambda'}^n + \sum_{\lambda'\in\wedge_h} u_{\lambda'}^n\langle\psi_\lambda, \psi_{\lambda'}''\rangle + \sum_{\lambda'\in\wedge_h} u_{\lambda'}^n\langle\psi_\lambda, \psi_{\lambda'}''''\rangle\right|.\end{aligned} \tag{5.31}$$

Let us now define

$$\delta_\lambda = \left|u_{\lambda'}^n + \sum_{\lambda'\in\wedge_h} u_{\lambda'}^n\langle\psi_\lambda, \psi_{\lambda'}''\rangle + \sum_{\lambda'\in\wedge_h} u_{\lambda'}^n\langle\psi_\lambda, \psi_{\lambda'}''''\rangle\right|.$$

The δ_λ is an ideal theoretical local error indicator. Because we have proved

$$[[u^{n+1} - u_h^{n+1}]]_n^2 \le C \sum_{\lambda \in \wedge_h} \delta_\lambda^2.$$

Proposition 5.5. *Let I_λ =supp ψ_λ, then we have*

$$2^{-j}|\delta_\lambda| \le C[[u - u_h]]_{n,I_\lambda}. \tag{5.32}$$

Proof. This can be proved straightaway using the definition of δ_λ. □

Unfortunately, the idea of using δ_k as an error indicator is not realistic due to the expensive computation and δ_λ is in general nonzero for all values $\lambda \in \Lambda \backslash \Lambda_h$, that is for an infinite numbers of values of λ. One needs to construct an indicator d_λ which retains the good properties of δ_λ, but is easier to handle in practice.

In order to define d_λ, we define the interactions of two indexes $\lambda = (j, k)$ and $\lambda^{'} = (j^{'}, k^{'})$. We define $j_m = \min\{j, j^{'}\}$ and $j_M = \max\{j, j^{'}\}$. Moreover, let $R = r - 2$, where we will suppose that $R > 2$. We set

$$v(\lambda, \lambda^{'}) = 2^{-(R/2)}|j - j^{'}|i(\lambda, \lambda^{'}),$$

with

$$i(\lambda, \lambda^{'}) = \begin{cases} 1 & \text{supp } \psi_\lambda \cap \text{supp } \psi_{\lambda'} \ne \Phi \\ 0 & \text{otherwise.} \end{cases} \tag{5.33}$$

For each λ, we define a neighborhood in Λ by $I_\lambda = \{\lambda^{'} : v(\lambda, \lambda^{'}) > \epsilon\}$, where ϵ is a given tolerance. Now, we can bound δ_λ with the sum of two components $\delta_\lambda \le d_\lambda + 2^{-j}|e_\lambda|$ with

$$d_\lambda = 2^{-j}\left| u_{\lambda'}^n + \sum_{\lambda' \in \wedge_h \cap I_\lambda} u_{\lambda'}^n \langle \psi_\lambda, \psi_{\lambda'}^{''} \rangle + \sum_{\lambda' \in \wedge_h \cap I_\lambda} u_{\lambda'}^n \langle \psi_\lambda, \psi_{\lambda'}^{''''} \rangle \right|$$

and

$$e_\lambda = \left| u_{\lambda'}^n + \sum_{\lambda' \in \wedge_h \backslash I_\lambda} u_{\lambda'}^n \langle \psi_\lambda, \psi_{\lambda'}^{''} \rangle + \sum_{\lambda' \in \wedge_h \backslash I_\lambda} u_{\lambda'}^n \langle \psi_\lambda, \psi_{\lambda'}^{''''} \rangle \right|.$$

The idea is to use d_λ as error indicator. In order to do this we need to show that the contribution of e_λ is negligible. Which can be proved by the following lemma.

Lemma 5.6. *We have*

$$\left(\sum_{\lambda \in \wedge \backslash \wedge_h} 2^{-2j}|e_\lambda|^2 \right)^{\frac{1}{2}} \le \epsilon[[u_h^n]]_n. \tag{5.34}$$

Proof. Do refer to Ref. 11. □

As a consequence we are now able to prove the following theorem

Theorem 5.7. *The following posteriori error estimate holds between the solution* $(u^n)_{0\le n\le N}$ *of the problem and the solution* $(u_h^n)_{0\le n\le N}$ *of the problem, for all* t_n, $1 \le n \le N$

$$[[u^{n+1} - u_h^{n+1}]]_n \le C\left(\sum_{\lambda\in\wedge\setminus\wedge_h} d_\lambda^2\right)^{\frac{1}{2}} + C\epsilon[[u_h^n]]_n \tag{5.35}$$
$$d_\lambda \le C([[u^{n+1} - u_h^{n+1}]]_{n,I_\lambda} + \epsilon[[u_h^n]]_n).$$

Proof.

$$\|r_h\|_{-1} \le C\left(\sum_{\lambda\in\wedge\setminus\wedge_h} \delta_\lambda^2\right)^{\frac{1}{2}} \le C\left(\sum_{\lambda\in\wedge\setminus\wedge_h} d_\lambda^2\right)^{\frac{1}{2}} + C\left(\sum_{\lambda\in\wedge\setminus\wedge_h} 2^{-2j}|e_\lambda|^2\right)^{\frac{1}{2}}$$
$$C\left(\sum_{\lambda\in\wedge\setminus\wedge_h} d_\lambda^2\right)^{\frac{1}{2}} + C\epsilon^2[[u_h^n]]_n$$

and since $2^{-j}|e_\lambda| \le C\epsilon[[u_h^n]]_n$, it gives the second part of (5.35). □

Acknowledgments

The first author would like to thank Dr. Nicholas Kevlahan for useful comments during the preparation of the paper.

References

1. A. Grossmann and J. Morlet, Decomposition of hardy functions in to square integrable wavelets of constant shape, *SIAM J. Math. Anal.* **15** (1984) 723–736.
2. N. K.-R. Kevlahan and Oleg V. Vasilyev, An adaptive wavelet collocation method for fluid-structure intercation at high reynolds numbers, *SIAM J. Sci. Comput.* **26**(6) (2005) 1894–1915.
3. O. V. Vasilyev and N. K.-R. Kevlahan, An adaptive multilevel wavelet collocation method for elliptic problems, *J. Comput. Phys.* **206** (2005) 412–431.
4. M. Mehra and N. K.-R. Kevlahan, An adaptive wavelet collocation method for partial differential equations on the sphere, to appear in *J. Comput. Phys.* (2008). http://dx.doi.org/10.1016/j.jcp.2008.02.004.
5. W. Dahman, A. J. Kurdila and P. Oswald (eds.), *Multiscale Wavelet Methods for Partial Differential Equations* (Academic Press, San Diego, CA, 1997).
6. M. Mehra and B. V. Rathish Kumar, Time accurate solution of advection-diffusion problems by wavelet-Taylor–Galerkin method, *Comm. Numer. Method. Eng.* **21** (2005) 316–326.
7. B. V. Rathish Kumar and M. Mehra, Time accurate fast wavelet Galerkin method for partial differential equation, *Numer. Methods Partial Differential Equations* **45** (2005) 479–503.
8. M. Mehra and B. V. Rathish Kumar, Error estimates for time accurate wavelet based schemes for hyperbolic partial differential equations, *Int. J. Wavelets Multiresolut. Inf. Process.* **5** (2007) 667–678.

9. K. Reiksson and C. Johnson, Adaptive finite element methods for parabolic problems. I. A linear model problem, *SIAM J. Numer. Anal.* **28** (1991) 43–77.
10. A. Bergam, C. Bernardi and Z. Mghazli, *A posteriori* analysis of the finite element discretization of some parabolic equations, *Math. Comput.* **10** (2004) 1117–1138.
11. S. Bertoluzza, *A posteriori* error estimates for the wavelet Galerkin method, *Appl. Math. Lett.* **8**(5) (1995) 1–6.
12. I. Daubechies, Orthonormal basis of compactly supported wavelets, *Comm. Pure Appl. Math.* **41** (1988) 906–966.
13. S. G. Mallat, Multiresolution approximations and wavelet orthonormal bases of L^R, *Trans. Amer. Math. Soc.* **315**(1) (1989) 69–87.
14. Y. Meyer, *Analysis at Urbana, I*: *Analysis in Function Spaces* (Cambridge University Press, 1989), pp. 256–365.
15. K. Yoshida, *Functional Analysis* (Springer-Verlag, 1974).
16. J. L. Lions and E. Magenes, *Non Homogeneous Boundary Value Problems and Applications* (Springer-Verlag, Berlin, 1972).
17. C. B. Jiang and M. Kawahara, The analysis of unsteady incompressible flows by a three-step finite element method, *Internat. J. Numer. Methods Fluids* **16** (1993) 793–811.
18. W. Dahman, S. Prossdorf and R. Schneider, Wavelet approximation methods for pseudodifferential equation I: Stability and convergence, *Math. Z.* **215** (1994) 583–620.
19. S. Bertoluzza, Adaptive wavelet collocation for the solution of steady state equations, *SPIE Proc.* **2491** (1995) 947–956.