Details of Minimizing χ^2 to Get Equations for Slope and Intercept of Best Fit Line

This is the derivation for the equations for the slope and intercept of the best fit line ($y = mx + b$) using the method of least squares fitting, mentioned in the **Least Squares Fitting with Excel** document.

$$\chi^2 = \sum (y_i - b - mx_i)^2$$

$$\begin{aligned}
\frac{\partial \chi^2}{\partial b} &= 2\left(\sum (y_i - b - mx_i)\right)(-1) \\
&= -2\left(\sum (y_i - b - mx_i)\right) \\
&= -2\sum y_i + 2\sum b + 2\sum mx_i \\
&= -2\sum y_i + 2Nb + 2m\sum x_i
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \chi^2}{\partial m} &= 2\left(\sum (y_i - b - mx_i)\right)(-x_i) \\
&= 2\left(\sum (x_i y_i - bx_i - mx_i^2)\right) \\
&= 2\sum x_i y_i - 2\sum bx_i - 2\sum mx_i^2 \\
&= 2\sum x_i y_i - 2b\sum x_i - 2m\sum x_i^2
\end{aligned}$$

Set $\frac{\partial \chi^2}{\partial b}$ and $\frac{\partial \chi^2}{\partial m}$ both equal to zero to minimize χ^2.

$$\begin{aligned}
&-2\sum y_i + 2Nb + 2m\sum x_i = 0 \\
&-\sum y_i + Nb + m\sum x_i = 0 \\
&Nb + m\sum x_i = \sum y_i
\end{aligned}$$

$$\begin{aligned}
&2\sum x_i y_i - 2b\sum x_i - 2m\sum x_i^2 = 0 \\
&\sum x_i y_i - b\sum x_i - m\sum x_i^2 = 0 \\
&b\sum x_i + m\sum x_i^2 = \sum x_i y_i
\end{aligned}$$

Solve the two equations simultaneously. You can do this by solving for one variable and substituting back into the other equation, but it is far easier to solve the equations in matrix form.

$$\begin{aligned}
&bN + m\sum x_i = \sum y_i \\
&b\sum x_i + m\sum x_i^2 = \sum x_i y_i
\end{aligned}$$

$$\begin{pmatrix} N & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix} \begin{pmatrix} b \\ m \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

$$\begin{pmatrix} N & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix}^{-1} \begin{pmatrix} N & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix} \begin{pmatrix} b \\ m \end{pmatrix} = \begin{pmatrix} N & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

$$\begin{pmatrix} b \\ m \end{pmatrix} = \begin{pmatrix} N & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

$$\begin{pmatrix} b \\ m \end{pmatrix} = \left(\frac{1}{N\sum x_i^2 - \left(\sum x_i\right)} \right) \begin{pmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & N \end{pmatrix} \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

$$\begin{pmatrix} b \\ m \end{pmatrix} = \left(\frac{1}{N\sum x_i^2 - \left(\sum x_i\right)} \right) \begin{pmatrix} \sum x_i^2 \sum y_i & -\sum x_i \sum x_i y_i \\ -\sum x_i \sum y_i & N\sum x_i y_i \end{pmatrix}$$

$$b = \frac{1}{N\sum x_i^2 - \left(\sum x_i\right)} \left(\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i \right)$$

$$= \frac{\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i}{N\sum x_i^2 - \left(\sum x_i\right)}$$

$$m = \frac{1}{N\sum x_i^2 - \left(\sum x_i\right)} \left(-\sum x_i \sum y_i + N\sum x_i y_i \right)$$

$$= \frac{N\sum x_i y_i - \sum x_i \sum y_i}{N\sum x_i^2 - \left(\sum x_i\right)}$$