Quantum Tutorial 1 (CYL110)

1. Equation $\rho(\upsilon,T)d\upsilon = \frac{8\pi h\upsilon^3}{c^3}\left(\frac{d\upsilon}{e^{\frac{h\upsilon}{kT}}-1}\right)$ expresses Planck's radiation law in terms of frequency. Express it in terms of wavelength.

2. Integrate Planck's distribution law over all frequencies to obtain the total energy emitted. What is its temperature dependence? What is this law known as? You will need to use the integral $\int_0^\infty \frac{x^3}{e^x-1} = \frac{\pi^4}{15}$.

3. Derive Stefan-Boltzmann law starting from Planck's radiation law (hint: $\int_0^\infty \frac{dx}{x^5\left(e^{\frac{1}{x}}-1\right)} = \frac{\pi^4}{15}$).

4. Derive Wien's law from Planck's radiation law.

5. We have seen that $\rho(\upsilon)d\upsilon = \frac{\langle\varepsilon\rangle N(\upsilon)d\upsilon}{V} = \frac{8\pi h\upsilon^3}{c^3}\left(\frac{d\upsilon}{e^{\frac{h\upsilon}{kT}}-1}\right)$, where <ε> is the average energy of the oscillators which are assumed to make up the walls of the cavity and which are in equilibrium with the radiation field inside the cavity, and N(ν)dν is the number of the modes of the radiation field in a cavity of volume V in the frequency range ν to ν +dν.
   (a) According to classical statistical mechanics, the average energy <ε> for systems in thermal equilibrium is $\langle\varepsilon\rangle = \frac{\iint \varepsilon e^{-\frac{\varepsilon}{kT}}dpdx}{\iint e^{-\frac{\varepsilon}{kT}}dpdx}$. For a simple harmonic oscillator $\varepsilon = p^2/2m + 1/2kx^2$. Evaluate <ε>.
   (b) If the energy of the oscillator can only take on the values $\varepsilon_i = ih\nu$, where i is an integer, then the integral in part (a) must be replaced by $\langle\varepsilon\rangle = \frac{\sum_{i=0}\varepsilon_i e^{-\frac{\varepsilon_i}{kT}}}{\sum_{i=0} e^{-\frac{\varepsilon_i}{kT}}}$. Evaluate <ε> (hint: $(1-y)^{-1} = \sum_{i=0}^{\infty} y^i$).

6. In the Davisson-Germer experiments, a single crystal of nickel was subjected to electron diffraction. The electron beam was accelerated through a potential difference of V volts to give it appropriate momentum. If the spacing between the lattice planes of the nickel is 1.7 angstrom, then calculate the minimum value of V needed to produce the appropriate de Broglie wavelength for the electrons.

7. When lithium is irradiated with light, one finds a stopping potential of 1.83 V for λ = 3000Å and 0.80 V for λ = 4000Å. From the data, calculate (a) Planck's constant, (b) threshold frequency, and (c) work function of Li.

8. Given $\hat{A} = d/dx$ and $\hat{E} = x^2$, show (a) $\hat{A}^2 f(x) \neq [\hat{A}f(x)]^2$ and (b) $\hat{A}\hat{E}f(x) \neq \hat{E}\hat{A}f(x)$ for arbitrary f(x).

9. Identify which of the following functions are eigenfunctions of the operators *d*/*dx* and d^2/dx^2: (a) e^{ikx} (b) cos kx (c) k (d) kx (e) e^{-ax2}. Give the corresponding eigenvalue where appropriate.

10. Find the eigenvalue in the following cases:

Â (operator)	f(x)	Eigenvalue
d^2/dx^2	cos ωx	

d/dt	exp(iωt)	
$\frac{d^2}{dx^2}+2\frac{d}{dx}+3$	exp(αx)	
$\partial/\partial y$	$x^2\exp(6y)$	

11. Write out the operator for $\hat{A}^2$ for $\hat{A}$ =
(a) $\frac{d^2}{dx^2}$ (b) $\frac{d}{dx}+x$ (c) $\frac{d^2}{dx^2}-2x\frac{d}{dx}+1$.

12. Find the eigenfunctions and eigenvalues of the operator d/dx.

13. In algebra it can be easily shown that $(P + Q)(P – Q) = P^2 – Q^2$. What is the value of (P + Q)(P – Q) if P and Q are operators? Under what conditions will this result be equal to $P^2 – Q^2$?

14. (a) Find [d/dx, x], $[z^3, d/dz]$, $[d^2/dx^2, ax^2 + bx + c]$, and $[a, a^\#]$ where $a = (x + ip)/\sqrt{2}$ and $a^\# = (x – ip)/\sqrt{2}$. (b) Determine whether the operators SQR and SQRT commute.

15. Evaluate the commutator $\left[\hat{A},\hat{B}\right]$, where $\hat{A}$ and $\hat{B}$ are given below:

	$\hat{A}$	$\hat{B}$
(a)	$\frac{d}{dx}-x$	$\frac{d}{dx}+x$
(b)	$\frac{d^2}{dx^2}-x$	$\frac{d}{dx}+x^2$

15. Normalize the following wavefunctions to unity: ($\int_0^\infty x^n e^{-ax}dx = n!/a^{n+1}$)

(a) sin(nπx/L) for the range 0 ≤ x ≤ L.
(b) c, a constant in the range –L ≤ x ≤ L.
(c) $\exp(–r/a_o)$ in 3-D.
(d) $x\exp(–r/2a_o)$ in 3-D.
(e) $(2-r/a_o)\exp(–r/2a_o)$ in 3-D.
(f) $r\sin\theta\cos\varphi\exp(–r/2a_o)$ in 3D

17. Which of the following candidates for wavefunctions are normalizable over the indicated intervals? Normalize those that can be normalized.

(a) $\exp\left(-x^2/2\right)$ (-∞, ∞) (b) e^x (0, ∞) (c) exp(iθ) (0, 2π) (d) xe^x (0, ∞) (e) $\exp\left[-\left(\frac{x^2+y^2}{2}\right)\right]$ (x,y: 0, ∞)

18. The wave function for a system can be written as $\psi(x)=\frac{1}{2}\phi_1(x)+\frac{1}{4}\phi_2(x)+\frac{3+i\sqrt{2}}{4}\phi_3(x)$ with $\phi_1(x)$, $\phi_2(x)$ and $\phi_3(x)$ being normalized eigenfunctions of the kinetic energy operator with eigenvalues E_1, $3E_1$ and $7E_1$ respectively. (a) Verify that ψ(x) is normalized. (b) What are the possible values of KE you will obtain in identically prepared systems. (c) What is the probability of measuring each of these eigenvalues? (d) What is the average value of kinetic energy that you would obtain from a large number of measurements.

[(c) 1/4, 1/16, 11/16 ; (d) $\langle E\rangle = 5.25E_1$ **]**