Chapter 1

Pressure, Potentials, And The Gradient

1.1 Field Theories in Physics

You are already familiar with the physics of forces, and you understand that an object can only be accelerated if a force acts on it. We have however avoided the fundamental questions of exactly how one object places a force upon an other object. You may respond, "How silly, a force is placed upon an object when another object pushes or pulls it." But there are forces for which the bodies do not need to be in contact, for example, gravity where it is possible for one planet to attract another even though the two are separated by a very large distance. This "action at a distance" has stymied many of the great minds, and the full understanding of this problem has come only within the last forty years. Unlike other theories, it is not the work of an isolated genius, but is the culmination of centuries of work by many people.

The question we will answer is how can one object place a force upon another without any apparent contact between the two whatsoever? Something must go between the two objects to carry the force, and we'll call it the field. We will direct our attention from the forces to the fields themselves. Note that this raises the level of abstraction considerably, but you should not forget that the reason for obtaining the field is to find the force. Always ask your self, "Now that I have found this field, what force would this field place upon my system." This course is the study of classical field theories. What properties must the fields have, and how do we describe these field?

It turns out, as you will see, that classical field theories do not fundamentally answer the question posed because one cannot touch or feel a classical electric field. More importantly, the electromagnetic fields are not conserved at all! They can be created and destroyed. Well, if they cannot be seen or felt and they are not conserved, do they exist? Might they be nothing more than a mathematical trick?

There is no way yet devised to tell whether the classical electromagnetic fields in fact do exist, but in any case, the question is not relevant because a modern theory with the unwieldy name of Quantum Electrodynamics (referred to as QED) has combined field theory, relativity, and quantum mechanics into a powerful theory that agrees with experiment to 14 decimal places! In QED the electric and magnetic field are combined to form discreet packets or quanta called photons. These photons sometimes behave as particles and they can be seen and counted by machines. In fact under special circumstances, it is possible for the human eye to "see" one photon.

In relativity mass and energy are equivalent, and it is possible, therefore, to create mass from photons, and conversely, to annihilate matter and antimatter to produce photons. This is done quite easily in modern particle accelerators. Relativity gives us a general conservation law called

Conservation of Mass–energy, and although the number of photons is not conserved and the mass is not conserved, the total amount of photons plus mass is conserved. Thus in modern physics, it is possible to show that photons, packets of electromagnetic fields, do exist and useful conservation laws have been formulated.

While QED may be too complicated to study here, we will introduce you to the basics behind classical field theory. Uses of classical fields range from thermodynamics and hydrodynamics to electromagnetics and computer science. They are found in virtually every field of science that uses sophisticated mathematics.

The fields are sometimes scalar and sometimes vector in nature so we will use the calculus you are already familiar with as well as vector calculus. Vector calculus is in some ways much like the calculus you have already grown to love, however in many ways, vector calculus is far different. Many of you will be taking vector calculus concurrently and will find that this course will overlap somewhat with vector calculus. We will not dwell on the mathematics per se; rather, we will study the physical properties of vector fields.

1.2 Path Integrals and the Gradient.

There are special vector fields that can be related to a scalar field. There is a very real advantage in doing so because scalar fields are far less complicated to work with than vector fields. A vector field may be derived from a scalar field any time the vector field is conservative. A conservative vector field $\vec{F}$ is required to have a zero path integral over any closed path , i.e.,

$$\oint \vec{F} \cdot d\vec{\ell} = 0. \qquad (1.1)$$

Note the conservative field definition follows from mechanics where the work is the path integral of the force:

$$W = \oint \vec{F} \cdot d\vec{\ell} \qquad (1.2)$$

The energy of a mechanical system is conserved when the work done around all closed paths is zero.

In general, a path integral is not like a regular integral, for example, the integral we all know and love deals with functions. For a function *g* whose derivative is *G*:

$$\frac{dg}{dx} = G \qquad (1.3)$$

the fundamental lemma of calculus states that

$$\int_{x_a}^{x_b} G\, dx = g(x_b) - g(x_a) \qquad (1.4)$$

where *g(x)* represents a well–defined function whose derivative exists. This is the integral developed in introductory calculus, but it is not the only definition of the integral. There are integrals called *path integrals* which have quite different properties. In general, a path integral does not define a function because the integral will depend on the path. For different paths the integral will return different results. In order for a path integral to return a function it must depend only on the end points as in the fundamental lemma (1.4). Then, a scalar field U will be related to the vector field $\vec{F}$ by

$$U(2) - U(1) = -\int_1^2 \vec{F} \cdot d\vec{\ell} \qquad (1.5)$$

The significance of the minus sign will be discussed in section 2.2. U is the potential energy if F is the force.

Because of the dependence on path, the path integral is also more complex than the fundamental lemma. For instance, the fundamental lemma can be inverted by differentiating as in equation (2–3), but the path integral needs a special operator to invert it. The need for a special operator is clear if we write a path integral in the Cartesian representation. Then

$$U(2) - U(1) = -\int_1^2 F_x\, dx - \int_1^2 F_y\, dy - \int_1^2 F_z\, dz \qquad (1.6)$$

where F_x, F_y, and F_z are the components of the vector field. In order for a relation like the fundamental lemma to exist, F_x must be the derivative of f with respect to x. Thus, for equation (1.6) to be satisfied,

$$F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z} \qquad (1.7)$$

We can satisfy all of the conditions (1.7) by defining the 'Del' operator. In the Cartesian representation

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \qquad (1.8)$$

An operator in nothing new to you. It simply represents a set of instructions for the performance of an operation on whatever lies to the right of the operator. For instance, the cross product sign $\times$ is an operator that tells you to use the determinant rule to multiply two vectors together, and d/dx instructs you to take the derivative with respect to x of anything that lies to the right.

Similarly, del operates on whatever resides to the right of it and basically follows the rules of vector multiplication. Because the del is an operator, it has no magnitude, and therefore, no meaning by itself. It only has meaning when it operates on something. Since del has no magnitude and is always a vector, we needn't bother writing an arrow over it.

We can transform del to other coordinate systems by looking at how the differentials behave. In spherical polar coordinates we have

$$\{dx, dy, dz\} \to \{dr, r d\theta, r \sin\theta\, d\phi\}$$

which when we transform del gives:

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi} \qquad (1.9)$$

As a reference note remember that transforming del works fine for del operating on a scalar, but becomes more complex when del operates on a vector. Expression (1.9) will not hold for the polar form when del operates on a vector.

Using del, equation (1.7) is an operation called the gradient. The Cartesian representation of the gradient is

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} \qquad (1.10)$$

where it's important to note that the result is a vector, and herein,lies the importance. It is possible to obtain a vector field $\nabla\mathcal{E}$ from a scalar field $\mathcal{E}$ by utilizing the gradient.

Example 1.1

$f(x,y,z) = 3x^2y^2 + 4x - 6y$ Find a) the gradient of f, and b) where the minimum value of f occurs.

Solution: a) $\nabla f = \hat{i}\frac{\partial}{\partial x}\left(3x^2y^2 + 4x\right) + \hat{j}\frac{\partial}{\partial y}\left(3x^2y^2 - 6y\right) + 0\hat{k} = \hat{\imath}(6xy^2 + 4) + \hat{\jmath}(6x^2y - 6)$.

b) The minimum in f occurs where its slope is zero or where $\nabla f = 0$. $\Rightarrow$

$$\hat{\imath}(6xy^2 + 4) + \hat{\jmath}(6x^2y - 6) = 0$$

In order for this to be true both the x and y components of the gradient ("slope") must independently go to zero.

$$6xy^2 + 4 = 0 \quad \text{and} \quad 6x^2y - 6 = 0$$

$$x = -\sqrt[3]{\frac{3}{2}} \quad \text{and} \quad y = \sqrt[3]{\frac{4}{9}}$$

1.3 The Gradient

Of what use can the gradient be? It looks horribly complicated, but as you will see, the power it gives us far outweighs its seeming complexity. Moreover, it is not just a mathematical curiosity but a tool that you will use often because its geometrical interpretation makes it so important. We'll start with a one dimensional function f(x) to illustrate some of the properties then proceed to more complex situations.

Example 1.2

Suppose a graph of $f(x)$ looks like that shown in Figure 1.1 where to the left of x_o the function has a positive slope, to the right a negative slope, and a maximum at x_o. Find where the gradient points in the positive x direction and where the gradient points in the negative x direction.

Solution: If $f(x)$ only depends on x, then its y and z partial derivatives are zero so that

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} \ .$$

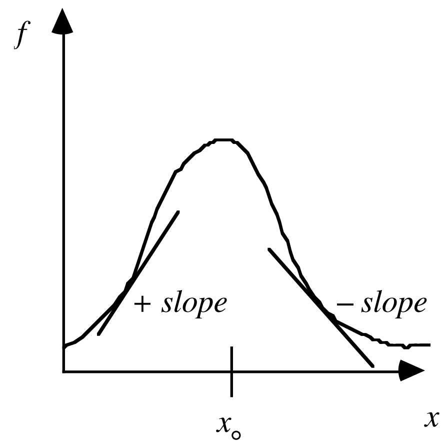
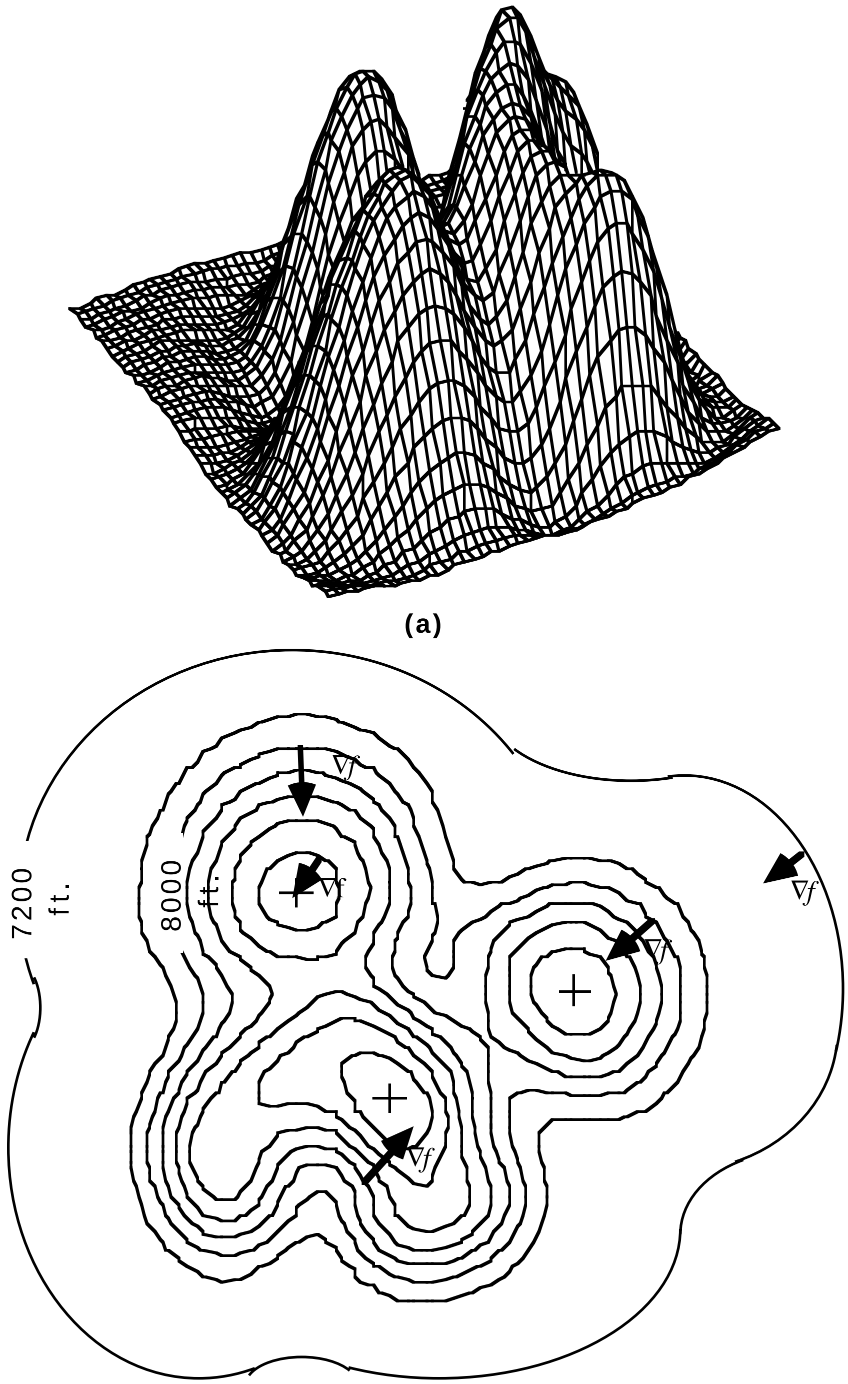


Figure 1.1
For a function of one dimension the gradient points towards the place where the function has a maximum.

Since $\partial f/\partial x$ is the slope, we see that the gradient is positive to the left of x_o and negative to the right. The gradient, thus, always will be a vector pointing towards x_o, i.e., the direction of the gradient will always point to the place where the function has a maximum, and the magnitude of the gradient will be the slope of the function.

In actuality the gradient is just the generalization of slope to functions of more than one variable. If we consider the function $f(x, y)$ to be a hill, the gradient will point towards the top of the hill, and its magnitude gives the steepness of the hill. Geologists and hikers implicitly employ the concept of gradient when reading topographic maps. When hiking in the mountains you quickly become aware of the effects of gravity, and it is because of these effects that we represent the locus of points at a given height as a contour line on a topographic map. Where contour lines are very close together, the country will be very steep because the height changes quickly in a short distance, and this means a large gradient. Remember that the gradient is the change in the scalar function divided by the change in distance.

Since the gradient is given by the change in the function, we can also determine the direction of the gradient at any point. Note that a function, by definition, does not change along a contour line; therefore, the change in the function must be perpendicular to the contour line. If we combine this with the fact that the gradient always points towards the top of the hill, then we have a very powerful and easy method for determining immediately from a contour plot where the gradient is large and also its direction. A plot of a function and its associated contour map is shown in Figure 1.2. The gradient is shown at a few representative points on the contour map. You should practice representing the gradient at other points



(a)

(b)

Figure 1.2

A function $f(x,y)$ in (a), and its contour map in (b). The two are not aligned. Can you find which places on the contour map correspond to places on the 3–D plot?

1.4 Gravity

First we will move away from a description based on the force by defining a gravitational field $\vec{g}$. Let's return to Newton's problem. Consider an apple starting from rest and accelerating freely under the influence of gravity. The force of the earth's attraction causes the apple to fall, but how specifically? Until the apple hits the ground, the earth does not touch the apple so how does the earth place a force on the apple? Something must go from the earth to the apple to cause it to fall, i.e., the earth must exude something that places a force on the apple. This something exuded by the earth we call the gravitational field. We can start by investigating the properties of the gravitational field.

To operationally define the gravitational field we note that since it is exuded by the earth, it should not depend on the mass of the apple. The force on the apple, however, does depend on the apple's mass so we can define the gravitational field by taking the force on the apple and dividing out the apple's mass. We must be careful because if the earth produces a gravitational field, so should the apple (albeit small compared to the earth's). So to get the field due to the earth, we should let the mass of the apple go to zero. Hence our operational definition of the field due to gravity is

$$\vec{g} = \lim_{m \to 0} \frac{\vec{F}}{m} \qquad (1.11)$$

where $\vec{F}$ is the force due to gravity on our test apple, and the limit means that the mass of the apple should be as small as possible. If we define … to be up, the force on an apple will be

$$\vec{F} = -mg\,\hat{j}$$

Of course it should be obvious why we have called the gravitational field $\vec{g}$: it is nothing less than the acceleration due to gravity. Note that the field $\vec{g}$ is the same for all objects a given distance from the earth's surface.

Newton's law of gravitation gives the force on our apple as

$$\vec{F} = -\frac{GMm}{r^2}\hat{r} \qquad (1.12)$$

where M is the mass of the earth , m is the mass of the apple, and r is the distance between the centers. $\hat{r}$ is unit vector that points radially away from the earth. Using our operational definition, the gravitational field a distance r from the center of the earth is

$$\vec{g} = -\frac{GM}{r^2}\hat{r} \qquad (1.13)$$

The next step is to define a gravitational scalar field which we will call the gravitational potential V. Using equation (1.5)

$$V(r_2) - V(r_1) = -\int_{r_1}^{r_2} \vec{g} \cdot d\vec{\ell} \qquad (1.14)$$

Example 1.3

Starting with the gravitational field (1.14), Find the gravitational potential an arbitrary distance r from the center of the the earth. Assume $r >$ Rearth.

Solution: Substituting equation (1.14) and taking r_1 to be at infinity where the potential due to the earth should be zero, and $r_2 = r$ an arbitrary distance from the center of the earth we obtain

$$V(r) - 0 = -\int_{\infty}^{r} \vec{g} \cdot d\vec{\ell}$$

Now if we integrate along a line coming radially in from infinity to *r*

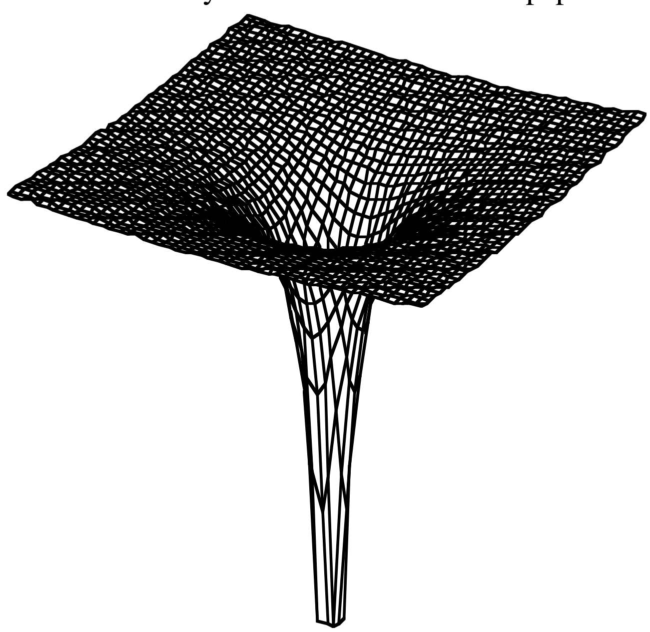
$$V(r) = -\int_{\infty}^{r} \left(-\frac{GM}{r^2} \right) dr = -\frac{GM}{r} + 0 \ . \qquad (1.15)$$

At sea level r is the radius of the earth so the gravitational potential is for all practical purposes constant.

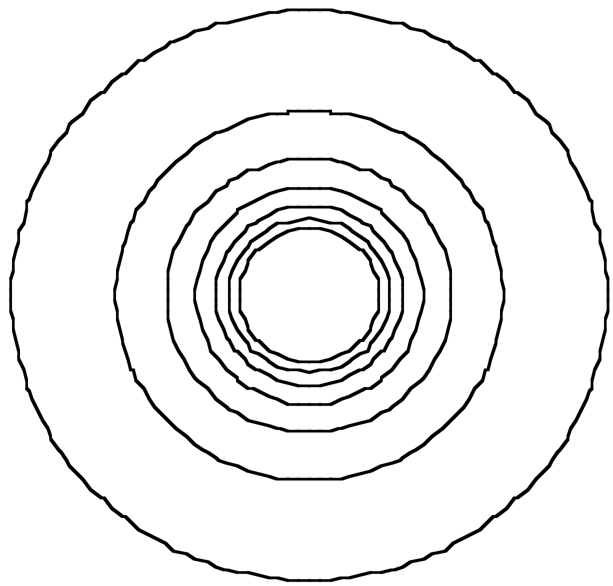
To find the potential energy *U* of an apple a distance r from the center of the earth merely multiply *V* by the mass of the apple *m*.

$$U = mV = -\frac{GMm}{r^2}. \qquad (1.16)$$

An inverse *r* potential is plotted in(1.16) (a) and the corresponding contour map of the potential is shown in Figure 1.3 (b). The plot has been truncated because of the singularity at r = 0. The contour lines are called equipotential lines. The contour map is just a 2 – D representation of a 3 – D situation. The equipotentials are not lines but surfaces that form concentric spheres about the mass. Our representations are limited by the medium of the 2–D paper this book is printed upon.



(a)



(b)
Figure 1.3
(a) an inverse r potential is plotted across the x-y plane. The singularity at the origin has been truncated. (b) A contour map of the inverse r potential in part a.

Figure 1.3 is very suggestive of a deep hole and this analogy gives us a way to conceptually interpret the potential. If our apple is placed in the potential, it will tend to roll down the potential hill towards the deep hole at the origin. The force on our apple will be greatest where the potential is steepest, i.e., where the gradient of the potential is greatest in magnitude. Also note that the force will be down the potential hill opposite to the gradient. Thus, the potential provides a visual means for displaying information about the fields. And as you might guess, the field lines1 can be found directly from the contour map by noting that the gravitational field is found from the potential by taking the gradient.

$$\vec{g} = -\nabla V . \tag{1.17}$$

As we saw in section 1.3, the gradient is always perpendicular to the contour lines which means we can map out the field lines by sketching lines that are always perpendicular to the equipotential lines. For the simple distribution of mass shown in Figure 1.3 the field lines form radial spokes

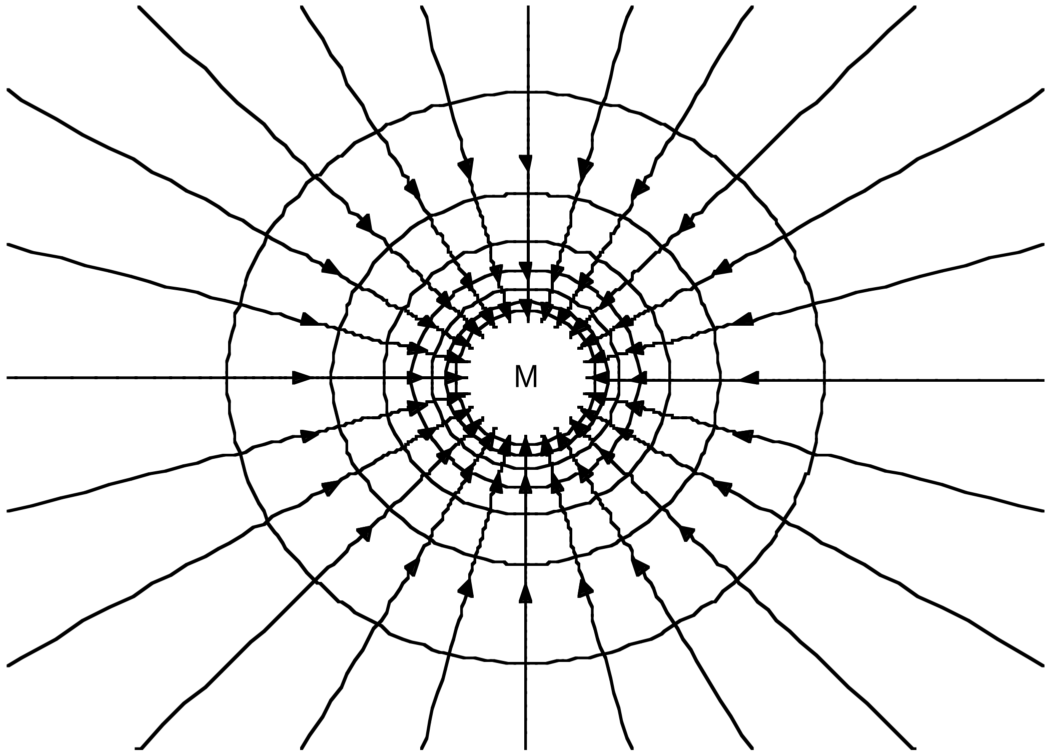


Figure 1.4
The gravitational field lines are always perpendicular to the equipotential lines.

towards the origin as shown in Figure 1.4. To understand more about the relation between scalar fields and vector fields it is instructive to look at fluids.

1.5 Fluids & Pressure.

Compared to solids fluids seem almost alive, magical. They flow, change form to accommodate the surroundings, produce gurgling sounds, and refract light to produce shimmer. There are few things that can match the majesty of a waterfall or the serenity of a deserted beach. What causes fluids to flow? As with solids, motions can only be produced by unbalanced forces so what is the nature of the forces in a fluid? The answer to our question can be quite complex so we will limit the discussion by only considering fluids with no viscosity. Such fluids are called inviscid which means there are no frictional effects. Slowly moving thin fluids, such as air and alcohol, approximate inviscid fluids; whereas, Prell shampoo and honey are examples of highly viscous fluids.

If the fluid is inviscid, there can be no drag or shear between the liquid and any surface it touches or between parts of the liquid that are in contact. This means the only force an inviscid fluid can exert against a boundary is normal to the boundary. The component of the force perpendicular to the boundary is caused by a pressure. We define the pressure to be the normal force $\vec{F} \cdot \hat{n}$ per unit area where $\hat{n}$ is the unit vector normal to the surface.

$$P = \frac{\vec{F} \cdot \hat{n}}{A} \quad . \qquad (1.18)$$

The SI unit for pressure is a N/m^2 which is called a Pascal.

It's important to realize that pressure always acts against a boundary. The boundary need not be the actual boundary between the fluid and an object. It can be an imagined boundary within the fluid which will allow us to analyze how pressure contributes to the motion of a fluid. In an inviscid fluid, pressure is a scalar; however, force is a vector so we might expect on vector considerations alone that the two are related by the gradient. We will now prove this. Consider a slab of fluid of thickness Δx that is a part of a much larger body of fluid as shown in figure Figure 1.5. The boundaries of the slab are imaginary and are just to make our discussion easier to visualize. On the left side of the slab the pressure is P and on the right side the pressure is P + ΔP. Note that the imbalance in the pressure will cause an imbalance in the forces. If ΔP is negative, such that, the pressure on the right side is less than the left, then the slab will move to the right.

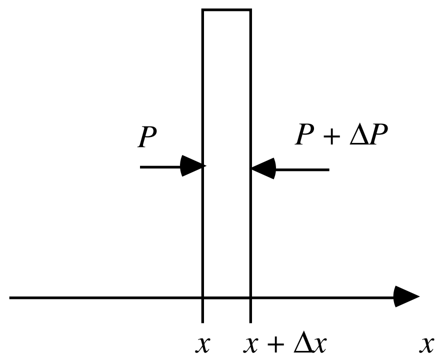


Figure 1.5
A thin slab of liquid experiencing a pressure gradient.

Summing the forces in the x direction on our slab we find

$$\Delta F_x = +PA - (P + \Delta P)A = -A\,\Delta P$$

where A is the area of the face of the slab normal to the force. Note that if we divide both sides by the volume

$$\Delta V = A\,\Delta x$$

so that

$$\frac{F_x}{V} = \frac{-AP}{A\Delta x}$$

Cancelling the area and taking the limit as Δx becomes infinitesimal:

$$f = \frac{F_x}{V} = -\frac{dP}{dx} \qquad (1.19)$$

We could have just as easily had pressure gradients in the y and z directions as well so the formula readily generalizes to three dimensions using the Del operator.

$$f = -\nabla P \qquad \textit{Force density} \qquad (1.20)$$

The minus sign in equation (1.20) is important because the gradient always points towards the maximum in the scalar function. The force on our slab of fluid will always be towards lower pressure which is opposite to the direction of the gradient.

Equation (1.20) gives the formal vector relation between the force and the pressure. At first the left–hand side might seem a bit strange, but it is only the force per unit volume—a force density. When dealing with fluids, it is natural to use the force per unit volume because although we chose an imaginary slab of the fluid, there is no real boundary separating that slab from the rest of the fluid. By dividing out the volume, we have rid ourselves of considering anything related specifically to the size of the slab. This can be seen by considering the gravitational force acting upon our slab.

$$\vec{F} = -mg\,\hat{j}$$

which depends explicitly on the mass of our particular slab. By using the force per unit volume we have

$$\vec{f} = -\rho g\,\hat{j} \qquad (1.21)$$

where ρ is the ordinary mass density. Note equation (1.21) is independent of the size of the imaginary slab which is convenient. Just as the force is the mass times the acceleration, the force density is the mass density times acceleration.

1.6 Hydrostatics

Hydrostatics is the study of fluids at rest. As in mechanics, just because the fluid is at rest does not mean there are no forces present; rather, the forces are in equilibrium. If no forces are present, there will be no pressure. Pressure is the result of forces acting on a fluid. Of the cases we encounter most often in every day life, gravity plays a major role in establishing the pressure within a fluid. To determine the pressure we must sum the forces acting upon a unit volume of the fluid as shown in Figure 1.6. There is the force due to gravity pulling it down and the pressure gradient holding it up. There are no other forces so these two forces must be in equilibrium.

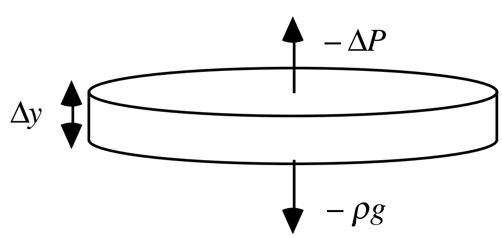


Figure 1.6
A thin slab of fluid in equilibrium with gravity.

Summing the force densities,

$$\Sigma f_y = -\nabla P - \rho g = 0 \qquad (1.22)$$

where the minus sign on the gradient is important because the force must point opposite to the gradient. Using

$$\nabla P = \frac{dP}{dy}\hat{j},$$

note that the pressure on the top must be less than the pressure on the bottom inorder for the slab of fluid to be in equilibrium with gravity, so the gradient points down and is in negative … direction. The equation that governs equilibrium is

$$-\frac{dP}{dy} - \rho g = 0. \qquad (1.23)$$

Equation(1.23) governs any fluid at rest in a uniform gravitational field g. For incompressible fluids, such as water, where the density is a constant the solution of (1.23) is simple, but for air and other gases the density may not be constant, and the solution becomes very tricky because the density is related to other factors, such as temperature. In fact, only for certain cses when the density satisfies a thermodynamic equation of state, such as the ideal gas law, does equation (1.23) even have a solution. If we allow ρ to arbitrarily vary with pressure, the solution often collapses to a point of infinite density. We will show how to solve equation (1.23) for an incompressible fluid, and the solution for an ideal gas will be left as a homework problem.

Example 1.4

The top surface of tank of water is at atmospheric pressure $P_\circ$. Find the pressure as a function of depth *h* below the top surface.

Solution: Since water is an incompressible fluid, the density is a constant, and we may solve equation (1.23) by separating variables.

$$dP = -\rho g\, dy \qquad (1.24)$$

Integrating *y* from [0, –*h*]

$$\int_{P_\circ}^{P} dP = -\rho g \int_0^{-h} dy$$

$$P = P_\circ + \rho g h \qquad (1.25)$$

where ρ and *g* are constants.

Equation (1.25) gives the pressure as a function of depth in an incompressible fluid whose top surface is at a pressure $P_\circ$. This pressure is determine by two things: 1) the forces acting upon the fluid (in this case gravity and the pressure gradient) and 2) the trivial equation of state ρ = constant.

Consider again our solution (2–29) for the pressure in an incompressible fluid. If the air pressure at the top of the tank is increased from $P_\circ$ to P', the pressure at a depth *h* becomes

$$P = P' + \rho g h,$$

that is, the change in pressure is passed undiminished to all points within the fluid. This is known as *Pascal's principal*.

Pascal's Principle

A change in pressure is transmitted undiminished to all parts of the fluid.

Your first reaction might be, "Wait, if the pressure changes, doesn't the force change, and then, won't the equilibrium will be upset?" The answer is no. The force due to the pressure depends only on the pressure gradient across the slab. The pressure on the top of the slab pushes down and it will be increased by the same amount as the pressure on the bottom of the slab which pushes up. Thus, since the force depends upon the gradient of the pressure, changing the pressure by any constant amount will not change the force.

Example 1.5

Pascal's principle is the foundation of all hydraulic apparatus. For the hydraulic lift shown below the circular piston 2 has a radius twice that of piston 1. If a downward force F_1 is applied to piston 1, find the force that must be applied to piston 2 to keep it stationary. Neglect the mass of the pistons.

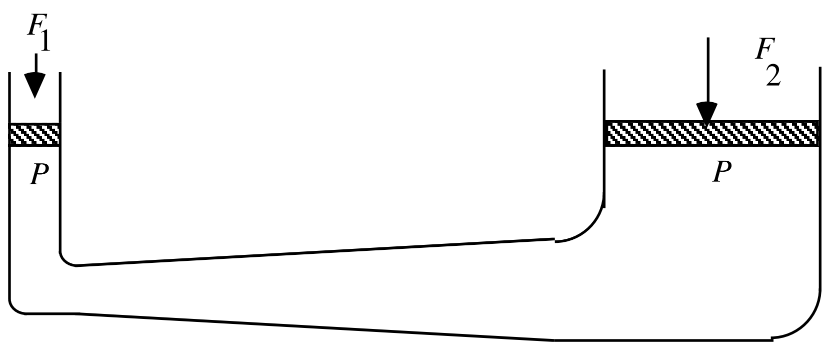


Figure 2.7
A simple hydraulic lift.

Solution: If the two pistons are at the same height, the pressure acting on both will be the same. The pressure will be determined by the forces acting upon the fluid. Summing the forces on piston 1,

$$\Sigma F_1 = PA_1 - P_\circ A_1 - F_1 = 0$$

where air pressure $P_\circ$ pushes down on piston 1, and the pressure P pushes up. Thus,

$$P = \frac{F_1}{A_1} + P_\circ . \qquad (1.26)$$

This same pressure acts up on piston 2, but it is distributed over a larger area, and will give rise to a larger force.

$$\Sigma F_2 = PA_2 - P_\circ A_2 - F_2 = 0$$

Substituting P in equation (1.26),

$$\frac{A_2}{A_1}F_1 - F_2 = 0$$

$$F_2 = \frac{r_2^{\,2}}{r_1^{\,2}}F_1 = 4F_1 \qquad (1.27)$$

There are two points to note. First, by making the area of the first piston very small compared to the second, it is quite possible to attain a very large mechanical advantage. Second, note that the air pressure contribution cancels out of (1.27) because it acts on both pistons.

We are in a position to understand why things float which follows from Archimedes' principle.

Archimedes' Principle

Any body wholly or partially submerged in a fluid is buoyed up by a force equal to the weight of the displaced fluid.

We will now prove Archimedes' principle by considering the floating object of mass m shown in Figure 1.7.

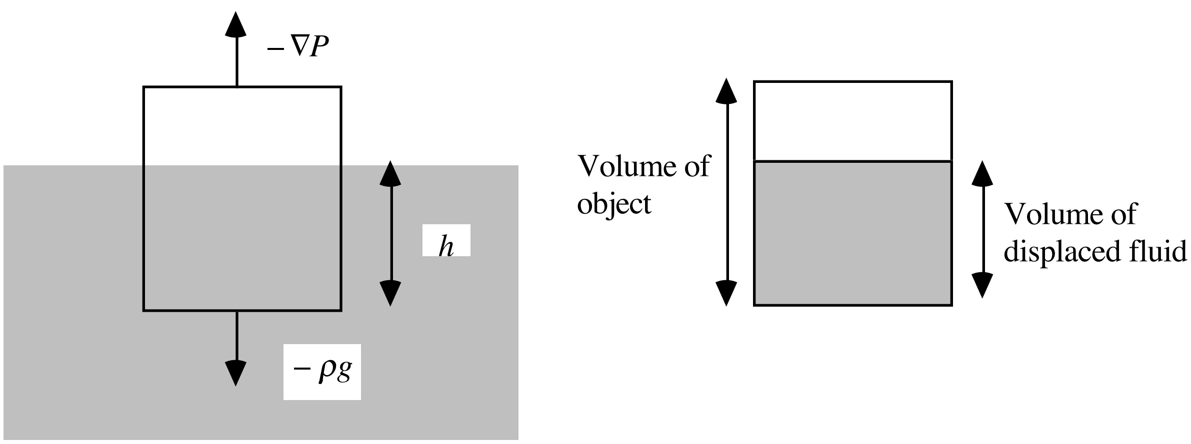


Figure 1.7
A partially floating object.

Summing the forces acting upon the object and noting that the pressure gradient occurs over a height h,

$$\sum F_y = \Delta PA - mg = 0$$

Using the volume of the displaced fluid $V_D = Ah$

$$\frac{\Delta P}{h}V_D - mg = 0 \qquad (1.28)$$

The pressure difference between the bottom and the top is

$$\Delta P = P_\circ + \rho_f gh - P_\circ = \rho_f gh$$

where ρ_f is the density of the fluid so equation (1.28) becomes

$$\rho_f g V_D - mg = 0. \qquad (2\text{–}36)$$

Note that $\rho_f g V_D$ is the buoyant force due to the pressure gradient, and since $m_f = \rho_f V_D$ is the mass of the displaced fluid we have proved Archimedes' principle. An object will float when the mass of the displaced fluid equals the mass of the object.

1.7 Gradients in Hydrodynamics

Unbalanced forces on a fluid will cause it to flow, to change in time, to be dynamic. Hydrodynamics is the study of fluids in motion. As we might guess from our previous discussion, any disturbance whose sole effect is to cause a pressure gradient will create an unbalanced force on the fluid causing it to flow. The direction of flow is opposite to the gradient, i.e., towards low pressure.The geometry of the flow can be depicted by plotting lines of constant pressure called *isobars*.

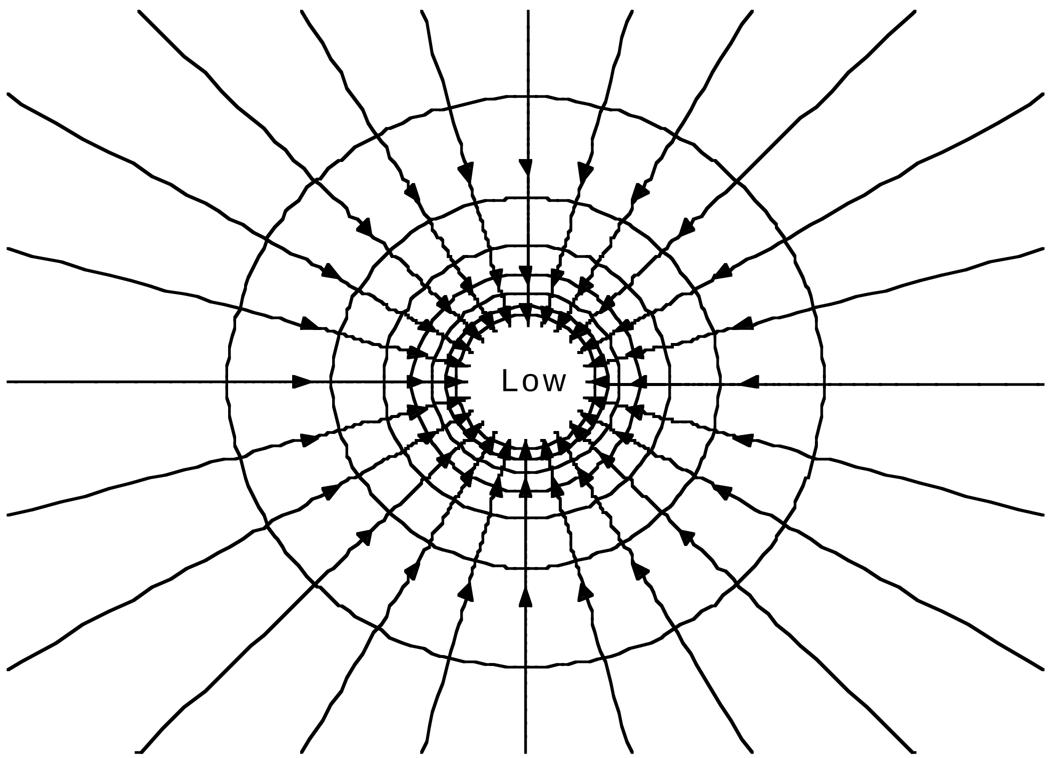


Figure 1.8
Isobars and lines of force for an ideal region of low pressure

Consider the cylindrical region of low pressure as shown in Figure 1.8. The vector field lines are lines of force in the fluid and are perpendicular to the isobars. Note how similar this is to the gravitational case shown in Figure 1.4. The theory for this type of fluid flow is referred to as *potential theory*. For low velocity flows without turbulence (called laminar flows) and other rotational effects (laminar flow in an irrotaional fluid), pieces of the fluid will approximately follow a trajectory along the lines of force. The force lines approximately map the velocity field for the flow. The field lines mapping the velocity are called *streamlines*. For steady state flow in which the velocity of the fluid at any point remains constant in time, the streamlines correspond to trajectories of small chunks of the fluid. But for time dependent flows, such as, when turbulence is present, the streamline pattern changes in time, and any given streamline will change long before a particle could travel along it, and so a streamline cannot represent the trajectory of a chunk of the fluid.

The shape of the isobars are determined by the cylindrical symmetry of the region. It is usually easy to deduce the isobars from the symmetry of the boundary conditions. Given one isobar, the next isobar out will look much like the previous; however, as we get further away from any region we loose detail until very far away the region will look like a point, and the isobars will be characteristic circles. Thus, isobars close to the region will mimic the shape of the region boundary, and then, will gradually soften into circles as the distance increases.

Example 1.6

Given the isobar

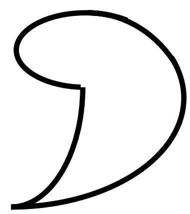


Figure 1.9

shown in Figure 1.9, sketch four more isobars.

Solution: The first isobar will look much like Figure 1.9, and further out the isobars will approach circles as the details of the source are lost.

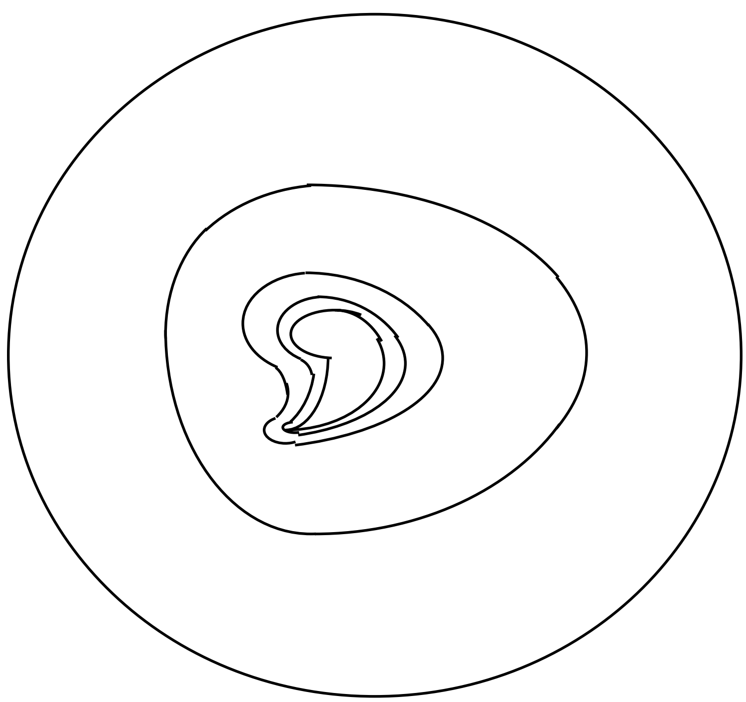


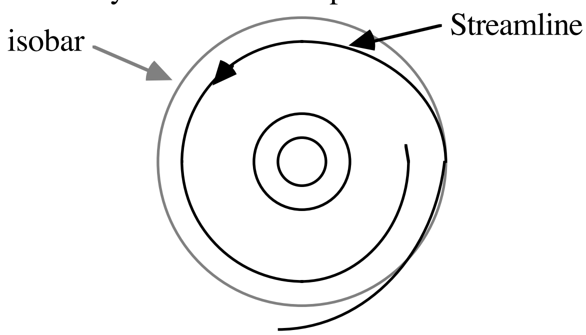
Figure 1.10

The inner isobars look like the first and the outer isobars look like circles.

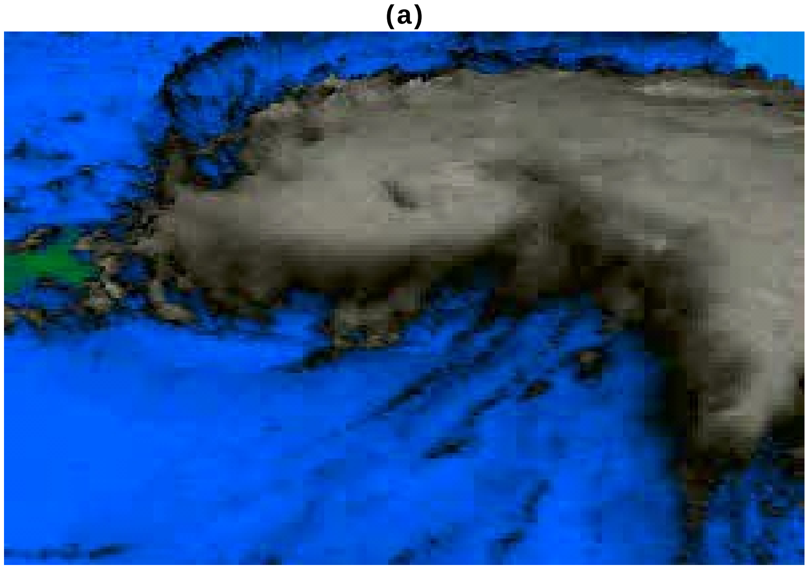
Once the isobars are drawn, it is relatively easy to obtain the streamlines. If the only force causing the flow is the pressure gradient, then the streamlines will point in the direction opposite to the pressure gradient. Since the pressure does not change along an isobar, the gradient must point in a direction perpendicular to the isobars and towards high pressure (uphill). The streamlines will also be perpendicular to the isobars but will flow towards low pressure.

It is important to realize the limitations of this picture we have developed because it is highly idealized. In many real systems there are nonconservative forces, such as friction between the fluid and a solid surface or between counter flowing layers of fluid, and in the earth's atmosphere the coriolis force, which invalidates the potential theory we have developed. If nonconservative forces are present, the streamlines will not necessarily be perpendicular to the isobars, and in fact, the exact opposite may be true.

Consider a hurricane in the northern hemisphere. As the air starts rushing in towards the low pressure center, the coriolis force causes a counter clockwise circulation about the low center. In such a case, the streamlines actually become almost parallel to the isobars



(a)



(b)

Figure 1.11

(a) In a hurricane the coriolis force causes the streamlines to be parallel to the isobars. (b) Hurricane Hortense.

as shown in Figure 1.12. The coriolis force explains why low pressure storms are so stable. The air cannot immediately rush in to dissipate the low pressure; instead, it gets bound up circulating about the low pressure, and the storm forms a stable low pressure cell. You can also see why hurricanes never hit the equator where the coriolis force is zero. Only when the force $\vec{F}$ is conservative are the streamlines perpendicular to the isobars, i.e., when

$$\oint \vec{F} \bullet d\vec{\ell} = 0 \tag{1.29}$$

$$\Rightarrow \vec{F}/V = -\nabla P. \tag{1.30}$$

We can now handle more complex situations. What would the velocity field look like for a low pressure region next to a high pressure region if there are no nonconservative forces? If we plot the pressure across the *x–y* plane, it is very suggestive of a mountain next to a valley

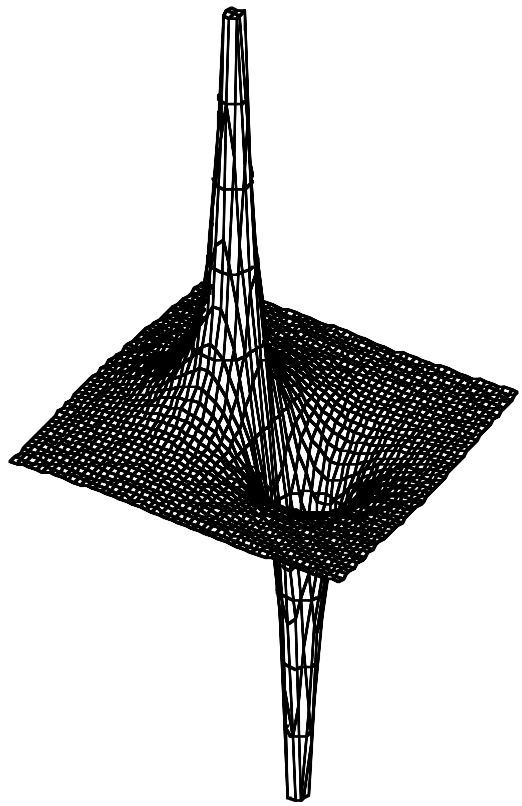


Figure 1.13
A plot of the pressure for a high pressure region next to a low pressure region.

as in Figure 1.13. The very high and very low isobars form concentric circles, but the isobar midway between the peak and the valley is a straight line. The isobars on either side of this straight line are flattened on the side facing the line.

Looking down from above we can make a contour representation of the isobars,

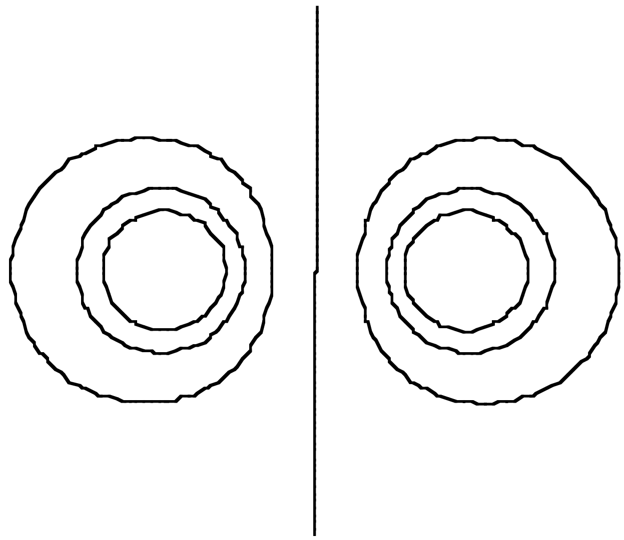


Figure 1.14
A contour map of a high pressure region next to a low pressure region.

The streamlines can be sketched in by drawing the lines perpendicular to the isobars.

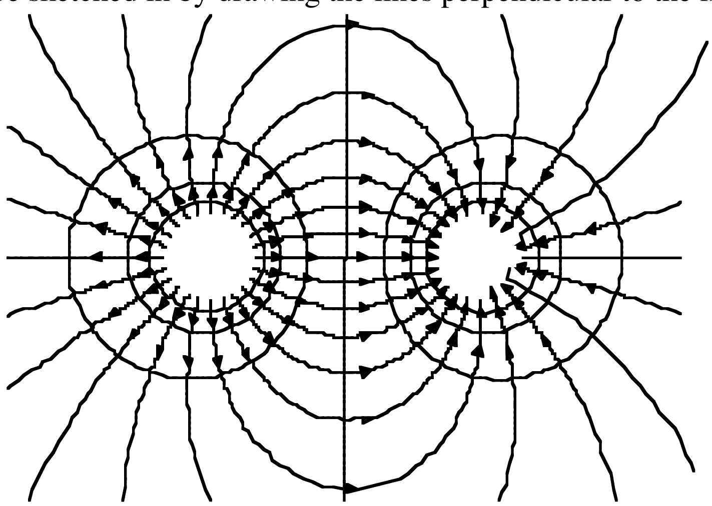


Figure 1.15
The streamlines are perpendicular to the isobars.

The flow will be greatest where the pressure surface is steepest, along a line joining the centers of the high and low pressure cells. You can see from Figure 1.13 that this regions forms a very steep cliff where the flow will be very large.

We have found graphical techniques which allow us to easily picture and interpret the gradient of a scalar field as a flow down a potential hill. It is quite amazing that the vector information on the flows can be obtained from a scalar pressure field that is much easier to work with than a vector field. Next we will consider even more powerful mathematical theorems to describe hydrodynamics.