1) Linear FO-ODE with constant coefficients:

$$\boxed{\frac{dy}{dx} = 2y+3}$$

$$\Rightarrow \frac{dy}{dx} = 2y+3$$

$$\frac{dy}{dx} = 2\left(y+\frac{3}{2}\right) \quad —① $$

$$\because \frac{d(y+3/2)}{dx} = \frac{dy}{dx} + \frac{d}{dx}(3/2)$$

$$\Rightarrow \frac{d(y+3/2)}{dx} = \frac{dy}{dx}$$ [$\because$ 3/2 is constant & hence not a function of $x$]

using the above relation, we can write

$$\frac{dy}{dx} = \frac{d(y+3/2)}{dx} = 2(y+3/2)$$

$$\Rightarrow \frac{d(y+3/2)}{dx} = 2(y+3/2)$$

Let $z = y+3/2$

$$\Rightarrow \frac{dz}{dx} = 2z$$

$$\Rightarrow \frac{dz}{z} = 2\,dx$$ [Assuming $z \neq 0$] —(*)

On integrating above expression, we get

$$\int \frac{dz}{z} = \int 2\,dx$$

$$\Rightarrow \ln(|z|) = 2x + c$$

$$\Rightarrow \ln\left(\left|y+\tfrac{3}{2}\right|\right) = 2x + c$$

$$\Rightarrow \left|y+\tfrac{3}{2}\right| = e^{2x+c}$$

$$\Rightarrow \left|y+\tfrac{3}{2}\right| = e^{2x}.e^{c}$$

$$\Rightarrow \left|y+\tfrac{3}{2}\right| = e^{c}.e^{2x}$$

$$\Rightarrow y+\tfrac{3}{2} = \pm e^{c}.e^{2x}$$

$\because e^{c} > 0$ so $\left.\begin{matrix} +e^{c} > 0 \\ -e^{c} < 0 \end{matrix}\right\} \pm e^{c} \in \mathbb{R} - \{0\}$

Let's put $\pm e^{c} = k$ (a constant)

$$\Rightarrow y+\tfrac{3}{2} = k\cdot e^{2x}$$

$$\Rightarrow y = ke^{2x} - \tfrac{3}{2}\ ;\ k \in \mathbb{R}$$

$$\boxed{y = ke^{2x} - \tfrac{3}{2}}\quad k \in \mathbb{R}$$

Note:- Since in (*) eqn, we assumed $z \neq 0$, hence $\pm e^{c} \in \mathbb{R} - \{0\}$ but for $z = 0$, the constant would take the value 0, hence for special case ~~so~~ $k \in \mathbb{R} - \{0\}$ we extended $z \neq 0$ to $k \in \mathbb{R}$ for any general $z$.

2) Initial Value Problem (IVP):

Linear FO.ODE (First order ODE) with constant coefficients:

$$\boxed{\frac{dy}{dx} = -3y + 1 \quad ; \quad y(0) = 1}$$

$$\Rightarrow \quad \frac{dy}{dx} + 3y = 1 \quad \text{—} ①$$

As explained in the lecture

$$I.F. = \mu = e^{\int 3dx} = k e^{3x}$$

and k is arbitrarily choosen as 1

so $\mu = e^{3x}$

Multiply with $\mu$ on both sides of eq ①

$$\Rightarrow \quad e^{3x}\frac{dy}{dx} + 3e^{3x}y = e^{3x}$$

$$\Rightarrow \quad \frac{d}{dx}(e^{3x}\cdot y) = e^{3x}$$

On integrating this expression, we get

$$\int \frac{d}{dx}(e^{3x}\cdot y)\,dx = \int e^{3x}dx$$

$$\Rightarrow \int d(e^{3x}\cdot y) = \int e^{3x}dx$$

$$\Rightarrow \quad e^{3x}\cdot y = \frac{e^{3x}}{3} + c$$

$$\Rightarrow \quad y = \frac{1}{3} + ce^{-3x}$$

$$\Rightarrow \quad y = ce^{-3x} + \frac{1}{3} \qquad \text{—} (*)$$

Now our job is to find that particular equation from the family of equations given above that satisfies our initial requirement i.e. our initial value $y(0) = 1$

Initial condition : $y(0) = 1$

on plugging this value in $(*)$, we get

$$1 = ce^{-3(0)} + \frac{1}{3}$$

$$1 = c + 1/3$$

$$\Rightarrow \quad \boxed{c = 2/3}$$

$$\Rightarrow \quad y = \frac{2}{3}e^{-3x} + \frac{1}{3}$$

or $\boxed{3y = 2e^{-3x} + 1}$

## 3) F.O - Linear ODE with variable coefficients :

$$\boxed{x\frac{dy}{dx} = -2y + 4x^2}$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{2}{x}\right)y + 4x \quad \text{--- (i)}$$

$$\Rightarrow \frac{dy}{dx} = a(x)y + b(x)$$

where $a(x) = \frac{-2}{x}$ & $b(x) = 4(x)$

Since coefficients are functions of $x$ & source function is linear in $y$ & we have $\frac{dy}{dx}$ hence F.O. - Linear ODE with variable coefficients

from (i)

$$\frac{dy}{dx} = -\left(\frac{2}{x}\right)y + 4x$$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{2}{x}\right)y = 4x \quad \text{--- (ii)}$$

so $I.F. = \mu = 1 \cdot \int e^{(2/x)dx} = 1 e^{2\ln x}$

$$\mu = (x^2)^{\ln e} \Rightarrow \boxed{\mu = x^2}$$

( We have calculated the $\mu(x)$ as explained in previous problem) so using (ii) & $\mu(x) = x^2$

we have

$$x^2\left[\frac{dy}{dx} + \frac{2}{x}y\right] = x^2[4x]$$

$$\Rightarrow \quad x^2\frac{dy}{dx} + (2x)y = 4x^3$$

$$\Rightarrow \quad \frac{d}{dx}\left[x^2y\right] = 4x^3$$

$$\Rightarrow \quad \int \frac{d}{dx}[x^2y]\,dx = \int 4x^3\,dx$$

$$\Rightarrow \quad \int d(x^2y) = \int 4x^3\,dx$$

$$\Rightarrow \quad x^2y = x^4 + c$$

$$\Rightarrow \quad y = x^2 + \frac{c}{x^2}$$

$$\boxed{y = \frac{c}{x^2} + x^2}$$

<u>Note:-</u>

If we convert this problem into IVP by imposing initial condition as $y(1) = 2$

then $2 = \frac{c}{1^2} + 1^2$

$$\Rightarrow \boxed{c = 1}$$

so $\boxed{y = x^2 + \frac{1}{x^2}}$

4) Non-Linear FO-ODE:

$$\boxed{\frac{dy}{dx} = y + 2y^5}$$

$$\Rightarrow \frac{dy}{dx} = (1)\,y + (2)\,y^5 \qquad \text{---(i)}$$

$$\Rightarrow \frac{dy}{dx} = p(x)\,y + q(x)\,y^n$$

where $p(x)$ turns out to be 1 } both constants
$q(x)$ turns out to be 2 }
& $n = 5$ which is clearly $\neq 0$ & $\neq 1$

$\Rightarrow$ This is a Non Linear FO-ODE with constant coefficients which is seperable

from (i)

$$\frac{dy}{dx} = y + 2y^5$$

divide the above equation by $y^5$

$$\frac{1}{y^5}\frac{dy}{dx} = \frac{1}{y^4} + 2 \qquad \text{--- (ii)}$$

Let $V = \dfrac{1}{y^4}$

$$\Rightarrow \frac{dV}{dx} = \frac{-4}{y^5}\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{y^5}\frac{dy}{dx} = \frac{-1}{4}\frac{dV}{dx} \qquad \text{--- (iii)}$$

from (ii) and (iii), we have

$$-\frac{1}{4}\frac{dv}{dx} = v+2$$

$$\frac{dv}{dx} = -4v-8$$

$\Rightarrow$ NL FO-ODE reduces to <u>linear</u> FO-ODE which is seperable

I.F. $\mu = e^{4x}$

so $e^{4x}\left[\frac{dv}{dx}+4v\right] = -8e^{4x}$

$$\frac{d}{dx}\left[e^{4x}.v\right] = -8e^{4x}$$

$\Rightarrow \int \frac{d}{dx}\left[e^{4x}\cdot v\right]dx = \int -8e^{4x}\,dx$

$\Rightarrow e^{4x}\cdot v = -2e^{4x}+c$

$\Rightarrow v = -2+ce^{-4x}$

also $\because v = \frac{1}{y^4}$

$\Rightarrow \frac{1}{y^4} = -2+ce^{-4x}$

$\Rightarrow \boxed{y = \pm\frac{1}{(ce^{-4x}-2)^{1/4}}}$

## (5) Euler Equation :

Show that

$$(x-y)\frac{dy}{dx} - 2y + 3x + \frac{y^2}{x} = 0 \quad -(i)$$

is an Euler homogeneous equation.

$$\Rightarrow \quad (x-y)\frac{dy}{dx} = 2y - 3x - \frac{y^2}{x}$$ {clearly this is non-seperable}

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2y - 3x - \frac{y^2}{x}}{x-y}$$

divide the numerator & denominator of the source function by x, i.e. multiply by $\frac{1}{x}$.

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\left(2y-3x-\frac{y^2}{x}\right)\left(\frac{1}{x}\right)}{(x-y)\,(1/x)}$$

$$\frac{dy}{dx} = \frac{2\left(\frac{y}{x}\right) - 3 - \left(\frac{y}{x}\right)^2}{1-\left(\frac{y}{x}\right)}$$

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

$\because$ source function can be reduced to a function of $\frac{y}{x}$, hence (i) is Euler homogeneous equation.

Let $v = \frac{y}{x}$

so $\frac{dy}{dx} = F\left(\frac{y}{x}\right) = F(v) = \frac{2v-3-v^2}{1-v}$ — (ii)

$\because$ $v = \frac{y}{x} \Rightarrow y = vx$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(vx)$

$= v\frac{d}{dx}(x) + x\frac{d}{dx}(v)$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ — (iii)

From (ii) & (iii) we have

$v + x\frac{dv}{dx} = \frac{2v-3-v^2}{1-v}$

$x\frac{dv}{dx} = \frac{2v-3-v^2-v+v^2}{(1-v)}$

$x\frac{dv}{dx} = \frac{v-3}{1-v} \Rightarrow \boxed{\frac{dv}{F(v)} = \frac{dx}{x}}$

The Original DE reduces to <u>seperable</u> form

$$\int \frac{dv}{\left(\frac{v-3}{1-v}\right)} = \int \frac{dx}{x}$$

Calculate above integral and put $v = \frac{y}{x}$ to finally get the solution.

6) Separable NL-F.O.ODE :

$$(4+y^3)\frac{dy}{dx} + (x^3-4x) = 0$$

⇒ It is FO because we have $\frac{dy}{dx}$

It is Non-Linear because we have $y^3$

so $$\frac{dy}{dx} = \frac{4x-x^3}{4+y^3}$$

or $$(4+y^3)\frac{dy}{dx} = 4x-x^3$$

$$(4+y^3)\,dy = (4x-x^3)\,dx$$

Clearly this shows that the ODE is <u>separable</u>

$$\int(4+y^3)\,dy = \int(4x-x^3)\,dx$$

$$4y+\frac{y^4}{4} = 2x^2-\frac{x^4}{4}+C$$

$$\Rightarrow \boxed{y^4+16y+x^4-8x^2=C}$$