Three-Dimensional Motion – Introduction

- Many engineering dynamics problems are solved using:
  - **Plane motion (2D)**
- However, modern applications require:
  - **Three-dimensional (3D) motion analysis**
- Key differences in 3D:
  - Vectors gain an additional **third component**
  - Angular quantities gain **two additional components**
  - Increased complexity in:
    - Kinematics
    - Kinetics
- Advantage:
  - Full power of **vector analysis** is utilized
- Important note:
  - Strong understanding of **plane motion** is essential
  - Many concepts extend directly from 2D to 3D
- Scope:
  - Introductory treatment of 3D rigid-body motion

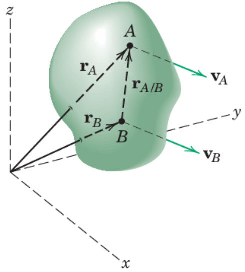
Translation in Three Dimensions

- A rigid body undergoes **translation** if:
  - All points move along **parallel paths**
- Types of translation:
  - **Rectilinear**: straight-line motion
  - **Curvilinear**: motion along congruent curves
- Key property:
  - Any line in the body remains **parallel to itself**
- Position relation:

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B}$$

- Since $\mathbf{r}_{A/B}$ = constant:

$$\mathbf{v}_A = \mathbf{v}_B, \quad \mathbf{a}_A = \mathbf{a}_B$$



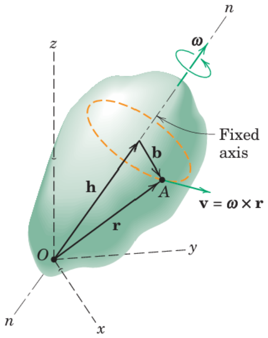
Fixed-Axis Rotation in 3D

- Rigid body rotates about a **fixed axis** n-n
- Angular velocity:
  - $\boldsymbol{\omega}$ is along the axis
  - Direction given by **right-hand rule**
  - Direction of $\boldsymbol{\omega}$ is **constant**
- Choose origin O on the axis
- Any point A not on axis:
  - Moves in a **circular path**
  - Plane is **perpendicular to axis**
- Velocity:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

- Using decomposition the above result can be seen:

$$\mathbf{r} = \mathbf{h} + \mathbf{b}, \quad \boldsymbol{\omega} \times \mathbf{h} = 0$$



- Acceleration obtained by differentiating:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

- Result:

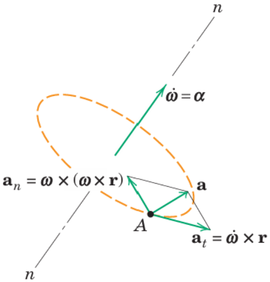
$$\mathbf{a} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

- Components:
  - **Tangential:**

$$\mathbf{a}_t = \dot{\boldsymbol{\omega}} \times \mathbf{r}$$

  - **Normal (centripetal):**

$$\mathbf{a}_n = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$



- Interpretation:
  - $\mathbf{a}_t$ $\rightarrow$ due to change in ω
  - $\mathbf{a}_n$ $\rightarrow$ always toward axis

Key Observations: Fixed-Axis Rotation

- Motion is equivalent to:
  - Circular motion about the axis
- Angular quantities:
  - $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are **collinear**
- Kinematic relations:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

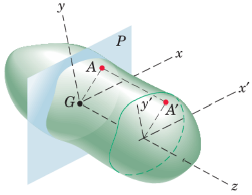
$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

- Special case:
  - If $\alpha = 0$:

$$\mathbf{a} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

Parallel-Plane Motion

- A rigid body undergoes **parallel-plane motion** when:
  - All points move in planes parallel to a fixed plane P
- This represents a **general form of plane motion**
- Reference plane:
  - Chosen through mass center G
  - Called the **plane of motion**
- Key idea:
  - Motion of any point (e.g., A')
  - is identical to corresponding point (A) in plane P
- Therefore:



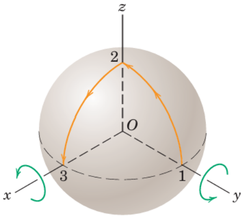
- 3D motion reduces to **equivalent 2D plane motion**

Interpretation of Parallel-Plane Motion

- Each point in the body:
  - Moves in its own plane
  - All such planes are **parallel**
- Correspondence:
  - $A \leftrightarrow A'$
  - Same motion projected onto reference plane
- Implication:
  - Full 3D motion described using:
    - Plane kinematics
- Practical advantage:
  - Reduces complexity of 3D motion
  - Allows use of:
    - Relative velocity
    - Instantaneous center
    - Plane motion equations

Rotation About a Fixed Point

- When a rigid body rotates about a **fixed point** O:
  - Angular velocity $\boldsymbol{\omega}$ **changes direction**
- This requires a more **general concept of rotation**
- **Rotation and Proper Vectors**
- Question:
  - Can rotations be treated as **vectors**?
  - Do they obey **parallelogram law**?
- Setup:
  - Solid sphere rotating about fixed point O
  - Axes x, y, z fixed in space
- Examine:

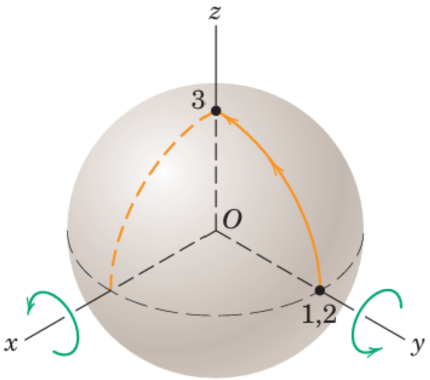


θ_x then θ_y

- Two successive 90° rotations
- Case 1:
  - Rotate about x-axis, then y-axis
  - Point moves: $1 \rightarrow 2 \rightarrow 3$

Finite Rotations are Not Commutative

- Case 2:
  - Rotate about y-axis, then x-axis
  - Point moves differently
- Result:
  - **Final positions are different**



θ_y then θ_x

Key Conclusion on Rotation Vectors

- Finite rotations:
  - Do **not obey parallelogram law**
  - Are **not commutative**
- Therefore:
  - Finite rotations **cannot be treated as proper vectors**
- Important implication:
  - Order of rotations **matters**
- Note:
  - Only **infinitesimal rotations** behave like vectors

Infinitesimal Rotations as Proper Vectors

- Unlike finite rotations:
  - **Infinitesimal rotations** obey vector laws
- Consider two small rotations:

$$d\boldsymbol{\theta}_1, \quad d\boldsymbol{\theta}_2$$

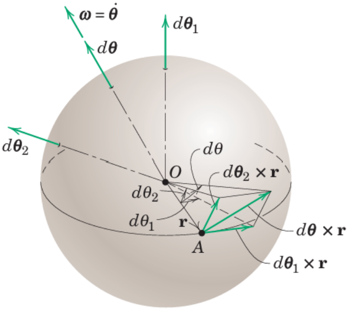
- Displacement of point A:
  - Due to $d\boldsymbol{\theta}_1$:

$$d\mathbf{r}_1 = d\boldsymbol{\theta}_1 \times \mathbf{r}$$

  - Due to $d\boldsymbol{\theta}_2$:

$$d\mathbf{r}_2 = d\boldsymbol{\theta}_2 \times \mathbf{r}$$

- Total displacement:



$$d\mathbf{r} = d\boldsymbol{\theta}_1 \times \mathbf{r} + d\boldsymbol{\theta}_2 \times \mathbf{r}$$

- Factor the expression:

$$d\mathbf{r} = (d\boldsymbol{\theta}_1 + d\boldsymbol{\theta}_2) \times \mathbf{r}$$

Vector Addition of Infinitesimal Rotations

- Therefore:

$$d\boldsymbol{\theta} = d\boldsymbol{\theta}_1 + d\boldsymbol{\theta}_2$$

- Key result:
  - Infinitesimal rotations are **commutative**
  - They obey **parallelogram law**

- Angular velocity:

$$\boldsymbol{\omega}_1 = \dot{\boldsymbol{\theta}}_1, \quad \boldsymbol{\omega}_2 = \dot{\boldsymbol{\theta}}_2$$

- Hence:

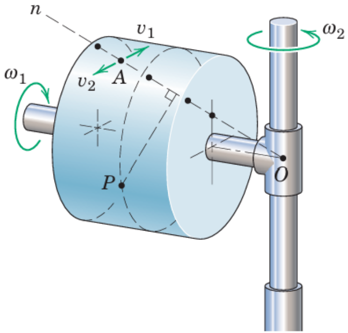
$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$$

- Interpretation:
  - At any instant, motion is equivalent to:
    - Rotation about a **single instantaneous axis**

- Consider a rigid body with:
  - Rotation ω_1 about its own shaft
  - Shaft rotating with ω_2 about a fixed axis
- At a given instant:
  - Some particles appear **stationary**
  - Others appear blurred (moving)
- Key observation:
  - A **line of points** has zero velocity
- This line defines:

Instantaneous axis of rotation $(O-n)$

- Example point A:
- Velocity components:



$$\mathbf{v}_1\ (\omega_1), \quad \mathbf{v}_2\ (\omega_2)$$

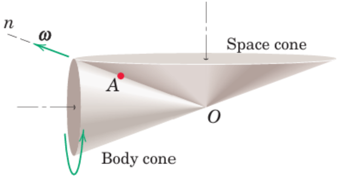
- Equal and opposite $\Rightarrow \mathbf{v} = 0$

Properties of Instantaneous Axis

- Points on axis O-n:
  - Have **zero velocity**
- All other points:
  - Move in **circular paths**
  - Planes are normal to the axis
- Motion interpretation:
  - Body undergoes **pure rotation about the instantaneous axis**
- Time evolution:
  - Axis changes position over time
  - Not fixed in the body
  - Not fixed in space
- Important conclusion:
  - General 3D motion $\equiv$ rotation about a **moving instantaneous axis**

Body and Space Cones

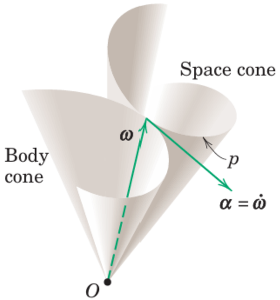
- Consider the instantaneous axis of rotation $O - A - n$
- Relative to the body:
  - The axis generates a **right-circular cone**
  - Called the **body cone**
- As motion continues:
  - The axis also sweeps a cone about the fixed vertical axis
  - Called the **space cone**
- Thus:
  - One cone attached to the body
  - One cone fixed in space



Rolling of Cones and Angular Velocity

- Key observation:
  - **Body cone rolls on space cone**
  - No slipping along their common generator
- Angular velocity:
  - ω lies along the **common element**
  - Defines the instantaneous axis
- Angular acceleration:

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$$



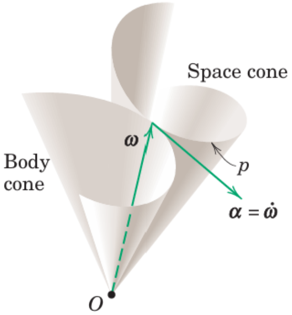
- General case:
  - Cones may not remain right-circular
  - But rolling relationship still holds
- Complex 3D rotation $\Rightarrow$ rolling of two cones
- Key takeaway:

Angular Acceleration in 3D Motion

- Angular acceleration:

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$$

- In **plane motion**:
  - α is scalar
  - Represents change in magnitude only
- In **3D motion**:
  - $\boldsymbol{\alpha}$ is a vector
  - Accounts for:
    - Change in magnitude of $\boldsymbol{\omega}$
    - Change in direction of $\boldsymbol{\omega}$
- Tip of $\boldsymbol{\omega}$ traces a space curve
  - $\boldsymbol{\alpha}$ is tangent to this curve



Angular Acceleration for Constant $|\omega|$

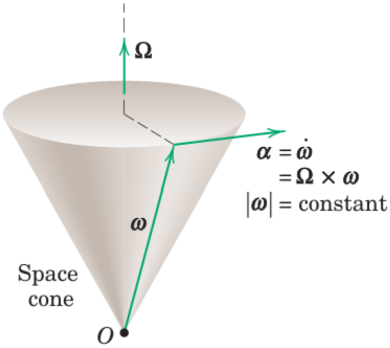
- If magnitude of $\boldsymbol{\omega}$ is constant:
  - Only direction changes
- Then:

$$\boldsymbol{\alpha} \perp \boldsymbol{\omega}$$

- Let $\boldsymbol{\Omega}$ = angular velocity of $\boldsymbol{\omega}$ (precession)
- Key relation:

$$\boxed{\boldsymbol{\alpha} = \boldsymbol{\Omega} \times \boldsymbol{\omega}}$$

- Interpretation:
  - $\boldsymbol{\omega}$ rotates about $\boldsymbol{\Omega}$
  - Forms a **space cone**

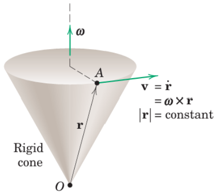


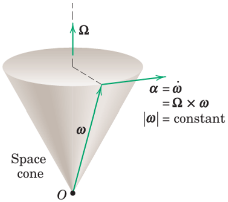
Geometric Analogy of Angular Acceleration

- Analogy with particle kinematics:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \quad \Longleftrightarrow \quad \boldsymbol{\alpha} = \boldsymbol{\Omega} \times \boldsymbol{\omega}$$

- Correspondence:
  - $\mathbf{r} \leftrightarrow \boldsymbol{\omega}$
  - $\mathbf{v} \leftrightarrow \boldsymbol{\alpha}$
  - $\boldsymbol{\omega} \leftrightarrow \boldsymbol{\Omega}$
- Insight:
  - Angular velocity behaves like a "position vector"
  - Angular acceleration behaves like "velocity"
- Thus:
  - Rotation of $\boldsymbol{\omega}$ produces $\boldsymbol{\alpha}$
  - Same structure as circular motion kinematics





Rotation about a Fixed Point: Kinematics

- For a rigid body rotating about a fixed point O:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

$$\mathbf{a} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

- These are identical to fixed-axis rotation relations
- **Key Difference (Fixed Axis vs Fixed Point):**
  - Fixed-axis rotation:
    - $\boldsymbol{\omega}$ direction is constant
    - $\boldsymbol{\alpha}$ only changes magnitude
  - Fixed-point rotation:
    - $\boldsymbol{\omega}$ changes direction
    - $\boldsymbol{\alpha}$ has:
    - Component along $\boldsymbol{\omega}$
    - Component normal to $\boldsymbol{\omega}$

Key Observations for Fixed-Point Rotation

- Points on instantaneous axis n–n:
  - Velocity $= 0$
  - Acceleration $\neq 0$ (if direction of $\boldsymbol{\omega}$ changes)
- For fixed-axis rotation:
  - Points on axis have:
    - Zero velocity
    - Zero acceleration
- **Important Conclusion:**
  - Rotation depends only on angular quantities:

$$\boldsymbol{\omega}, \quad \boldsymbol{\alpha}$$

  - Independent of motion of the reference point
  - Same expressions apply even if the point O moves

Sample Problem: 3D Rotation of a Disk

Given:

- Disk rotates about its own axis at:

$$\omega_0 = 120 \text{ rev/min}$$

- Assembly rotates about vertical Z-axis at:

$$N = 60 \text{ rev/min}$$

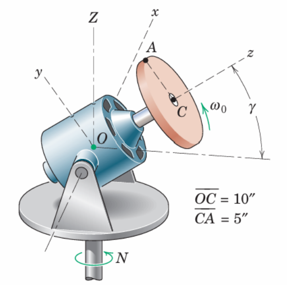
- Fixed inclination angle:

$$\gamma = 30^\circ$$

- Geometry:

$$OC = 10 \text{ in}, \quad CA = 5 \text{ in}$$

- Initially, the base is at rest



Find:

1. Angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$ of the disk
2. Space cone and body cone
3. Velocity and acceleration of

Solution: Angular Velocity

- $\omega_0 = 4\pi$ rad/s, $\quad \Omega = 2\pi$ rad/s
- Vertical axis:

$$\mathbf{K} = \mathbf{j}\cos\gamma + \mathbf{k}\sin\gamma$$

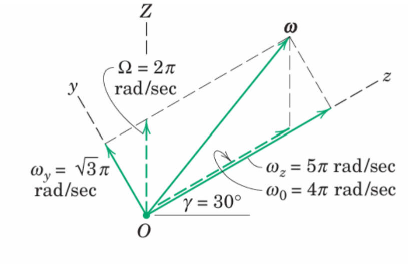
- Total angular velocity:

$$\boldsymbol{\omega} = \omega_0 \mathbf{k} + \Omega \mathbf{K}$$

$$= (\Omega\cos\gamma)\mathbf{j} + (\omega_0 + \Omega\sin\gamma)\mathbf{k}$$

- For $\gamma = 30^\circ$:

$$\boxed{\boldsymbol{\omega} = \pi(\sqrt{3}\mathbf{j} + 5\mathbf{k})}$$



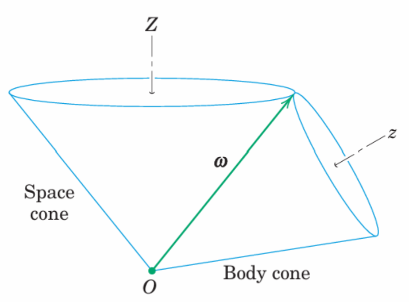
Angular Acceleration and Cones

- Angular acceleration:

$$\boldsymbol{\alpha} = \boldsymbol{\Omega} \times \boldsymbol{\omega}$$

$$\boldsymbol{\alpha} = (\Omega\omega_0 \cos\gamma)\mathbf{i}$$

$$\boxed{\boldsymbol{\alpha} = 68.4\,\mathbf{i}\ \text{rad/s}^2}$$



- **Geometric Interpretation:**
  - $\boldsymbol{\omega}$ traces:
    - Space cone (about Z)
    - Body cone (attached to disk)
  - $\boldsymbol{\omega}$ = common generator

General Motion: Translating Reference Axes

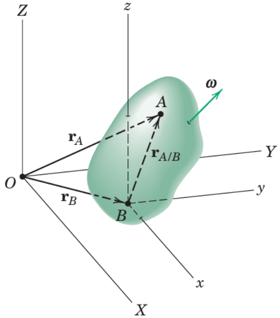
- General 3D motion is analyzed using:
  - Relative motion principles
  - Translating and rotating reference frames
- Choose point B as origin of translating frame x-y-z
- For any point A:

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B}$$

- Velocity relation:

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

- Acceleration relation:



$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$$

- Valid for translating (non-rotating) reference axes

General Motion: Translation + Rotation about B

- Distance AB remains constant $\Rightarrow$ rigid body
- Motion can be viewed as:
  - Translation of point B
  - Rotation of body about B
- Relative-motion terms represent rotation about B
- **Velocity relation:**

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{A/B}$$

- **Acceleration relation:**

$$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}} \times \mathbf{r}_{A/B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/B})$$

Reference Point Choice and Link Constraints

- Reference point B is arbitrary
  - Chosen for convenience
  - Often a point with known motion
- Using point A instead:

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$
$$\mathbf{a}_B = \mathbf{a}_A + \dot{\boldsymbol{\omega}} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A})$$

- Note:

$$\mathbf{r}_{B/A} = -\mathbf{r}_{A/B}$$

- $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$ are independent of reference point

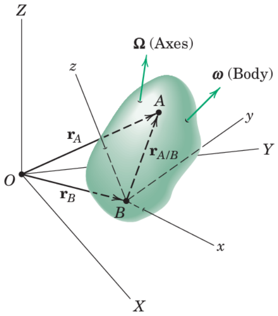
- **Rigid link constraint:**
  - Only angular motion normal to link matters

$$\boldsymbol{\omega}_n \cdot \mathbf{r}_{A/B} = 0$$

  - Similarly for angular acceleration:

Rotating Reference Axes

- General 3D motion requires:
  - Translating + rotating reference axes
- Reference frame:
  - Origin at point B
  - Axes rotate with angular velocity $\boldsymbol{\Omega}$
- Note:
  - $\boldsymbol{\Omega}$ (axes) $\neq \boldsymbol{\omega}$ (body)
- Extension to 3D:
  - Same relations as plane motion
  - Include z-components



Time Derivatives in Rotating Axes

- Unit vectors rotate with angular velocity $\mathbf{\Omega}$
- Time derivatives:

$$\dot{\mathbf{i}} = \mathbf{\Omega} \times \mathbf{i}$$
$$\dot{\mathbf{j}} = \mathbf{\Omega} \times \mathbf{j}$$
$$\dot{\mathbf{k}} = \mathbf{\Omega} \times \mathbf{k}$$

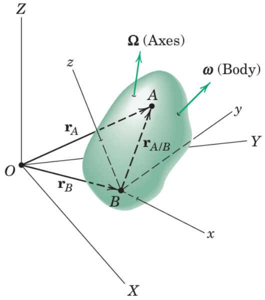
- Interpretation:
  - Change in direction of unit vectors due to rotation
  - Basis for velocity and acceleration transformations
- Key idea:
  - Time derivative in rotating frame introduces $\mathbf{\Omega} \times (\cdot)$ terms

Velocity in Rotating Reference Axes

- Absolute velocity of point A:

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\Omega} \times \mathbf{r}_{A/B} + \mathbf{v}_{\text{rel}}$$

- Terms:
  - $\mathbf{v}_B$: velocity of reference point B
  - $\boldsymbol{\Omega} \times \mathbf{r}_{A/B}$: due to rotation of axes
  - $\mathbf{v}_{\text{rel}}$: motion relative to rotating frame



- Relative velocity:

$$\mathbf{v}_{\text{rel}} = \dot{x}\,\mathbf{i} + \dot{y}\,\mathbf{j} + \dot{z}\,\mathbf{k}$$

- Key idea:
  - Observed motion $=$ transport motion $+$ relative motion

Acceleration in Rotating Reference Axes

- Absolute acceleration of point A:

$$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}) + 2\,\boldsymbol{\Omega} \times \mathbf{v}_{\mathsf{rel}} + \mathbf{a}_{\mathsf{rel}}$$

  - Terms:
    - $\dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B}$: angular acceleration
    - $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B})$: centripetal
    - $2\,\boldsymbol{\Omega} \times \mathbf{v}_{\mathsf{rel}}$: **Coriolis acceleration**
    - $\mathbf{a}_{\mathsf{rel}}$: relative acceleration
  - Relative acceleration:

$$\mathbf{a}_{\mathsf{rel}} = \ddot{x}\,\mathbf{i} + \ddot{y}\,\mathbf{j} + \ddot{z}\,\mathbf{k}$$

- Special case (axes fixed to body):
  - $\boldsymbol{\Omega} = \boldsymbol{\omega}$
  - $\mathbf{v}_{\mathsf{rel}} = 0, \;\; \mathbf{a}_{\mathsf{rel}} = 0$

$\Rightarrow$ reduces to rigid-body motion equations

Time Derivative in Rotating Reference Frame

- Relationship between derivatives in fixed $(X-Y-Z)$ and rotating $(x-y-z)$ frames:

$$\left(\frac{d\mathbf{V}}{dt}\right)_{XYZ} = \left(\frac{d\mathbf{V}}{dt}\right)_{xyz} + \mathbf{\Omega} \times \mathbf{V}$$

- Interpretation:
  - First term: change observed in rotating frame
  - Second term: change due to rotation of axes
- Apply to position vector:

$$\mathbf{r}_{A/B} = \mathbf{r}_A - \mathbf{r}_B$$

$$\left(\frac{d\mathbf{r}_A}{dt}\right)_{XYZ} = \left(\frac{d\mathbf{r}_B}{dt}\right)_{XYZ} + \left(\frac{d\mathbf{r}_{A/B}}{dt}\right)_{xyz} + \mathbf{\Omega} \times \mathbf{r}_{A/B}$$

- Result (velocity equation):

$$\boxed{\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{\text{rel}} + \mathbf{\Omega} \times \mathbf{r}_{A/B}}$$

Second Derivative and Acceleration Relation

- Apply operator again to obtain second derivative:

$$\left(\frac{d^2\mathbf{V}}{dt^2}\right)_{XYZ} = \left(\frac{d^2\mathbf{V}}{dt^2}\right)_{xyz} + \dot{\mathbf{\Omega}} \times \mathbf{V} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{V}) + 2\,\mathbf{\Omega} \times \left(\frac{d\mathbf{V}}{dt}\right)_{xyz}$$

- Key components:
  - $\dot{\mathbf{\Omega}} \times \mathbf{V} \rightarrow$ angular acceleration term
  - $\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{V}) \rightarrow$ centripetal term
  - $2\mathbf{\Omega} \times \left(\frac{d\mathbf{V}}{dt}\right)_{xyz} \rightarrow$ **Coriolis term**

- Important note:
  - This form directly leads to the general acceleration equation (Eq. 7/6)
  - Same structure applies for $\mathbf{a}_{A/B} = \mathbf{a}_A - \mathbf{a}_B$

Sample Problem 7/5

Given:

- Motor housing rotates about Z-axis:

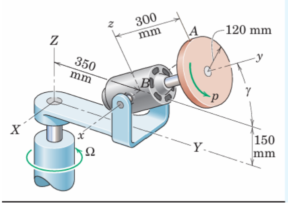
$$\Omega = 3 \text{ rad/s}$$

- Shaft and disk spin relative to housing:

$$p = 8 \text{ rad/s}$$

- Fixed angle:

$$\gamma = 30^\circ$$



Find:

- Velocity of point A
- Acceleration of point A
- Angular acceleration of the disk (α)

Solution: Velocity of Point A

Reference frames:

- Rotating axes $x - y - z$ attached to motor housing
- Fixed axes $X - Y - Z$ with unit vectors $\mathbf{I}, \mathbf{J}, \mathbf{K}$
- Rotating axes use unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$

Angular velocity of axes:

$$\boldsymbol{\Omega} = \Omega\mathbf{K} = 3\mathbf{K} \text{ rad/s}$$

Velocity equation:

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\Omega} \times \mathbf{r}_{A/B} + \mathbf{v}_{\text{rel}}$$

Compute terms:

$$\mathbf{v}_B = \boldsymbol{\Omega} \times \mathbf{r}_B = 3\mathbf{K} \times 0.350\mathbf{J} = -1.05\mathbf{i}$$

$$\boldsymbol{\Omega} \times \mathbf{r}_{A/B} = 3\mathbf{K} \times (0.300\mathbf{j} + 0.120\mathbf{k}) = -0.599\mathbf{i}$$

$$\mathbf{v}_{\text{rel}} = \mathbf{p} \times \mathbf{r}_{A/B} = 8\mathbf{j} \times (0.300\mathbf{j} + 0.120\mathbf{k}) = 0.960\mathbf{i}$$

Result:

$$\boxed{\mathbf{v}_A = -0.689\mathbf{i} \text{ m/s}}$$

Solution: Acceleration of Point A

Acceleration equation:

$$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}) + 2\boldsymbol{\Omega} \times \mathbf{v}_{\text{rel}} + \mathbf{a}_{\text{rel}}$$

Terms:

$$\mathbf{a}_B = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_B) = -2.73\mathbf{j} + 1.575\mathbf{k}$$

$$\dot{\boldsymbol{\Omega}} = 0$$

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}) = -1.557\mathbf{j} + 0.899\mathbf{k}$$

$$2\boldsymbol{\Omega} \times \mathbf{v}_{\text{rel}} = 4.99\mathbf{j} - 2.88\mathbf{k}$$

$$\mathbf{a}_{\text{rel}} = \mathbf{p} \times (\mathbf{p} \times \mathbf{r}_{A/B}) = -7.68\mathbf{k}$$

Result:

$$\boxed{\mathbf{a}_A = 0.703\mathbf{j} - 8.09\mathbf{k} \text{ m/s}^2}$$

$$|\mathbf{a}_A| = 8.12 \text{ m/s}^2$$

Solution: Angular Acceleration of Disk

Since precession is steady:

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \boldsymbol{\Omega} \times \boldsymbol{\omega}$$

$$\boldsymbol{\omega} = 3\mathbf{K} + 8\mathbf{j}$$

$$\boldsymbol{\alpha} = 3\mathbf{K} \times (3\mathbf{K} + 8\mathbf{j})$$

$$= 0 + (-24\cos 30^\circ)\mathbf{i}$$

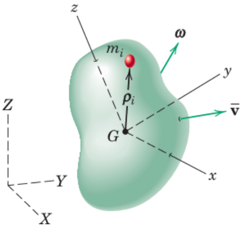
Result:

$$\boxed{\boldsymbol{\alpha} = -20.8\mathbf{i} \text{ rad/s}^2}$$

Angular Momentum of a Rigid Body

Concept:

- Extension of Newton's 2nd law to rotational motion
- More complex in 3D due to additional components



Setup:

- Axes $x-y-z$ attached to body at mass center G
- Angular velocity: $\boldsymbol{\omega}$
- Position of particle: $\boldsymbol{\rho}_i$

Angular momentum about G:

$$\mathbf{H}_G = \sum \boldsymbol{\rho}_i \times m_i \mathbf{v}_i$$

Angular Momentum About Mass Center

Substitute velocity:

$$\mathbf{H}_G = \sum \boldsymbol{\rho}_i \times m_i \left(\bar{\mathbf{v}} + \boldsymbol{\omega} \times \boldsymbol{\rho}_i \right)$$

Expand:

$$\mathbf{H}_G = -\bar{\mathbf{v}} \times \sum m_i \boldsymbol{\rho}_i + \sum \left[\boldsymbol{\rho}_i \times m_i (\boldsymbol{\omega} \times \boldsymbol{\rho}_i) \right]$$

Key result:

$$\sum m_i \boldsymbol{\rho}_i = 0 \quad \text{(about } G\text{)}$$

Thus:

$$\boxed{\mathbf{H}_G = \int \left[\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) \right] dm}$$

Angular Momentum About a Fixed Point

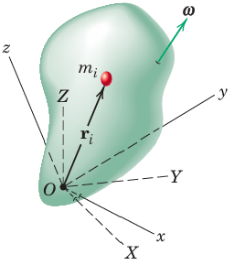
For rotation about point O:

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i$$

Angular momentum:

$$\mathbf{H}_O = \sum \mathbf{r}_i \times m_i \mathbf{v}_i$$

$$\boxed{\mathbf{H}_O = \int [\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})]\, dm}$$



Notes:

- Valid for rigid body rotation about fixed point
- Foundation for moment equations in 3D motion

Moments and Products of Inertia

Key Idea:

- Angular momentum expressions have identical form
- Position vectors:

$$\boldsymbol{\rho}_i = \mathbf{r}_i = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Approach:

- Expand triple vector product:

$$\mathbf{H} = \int \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})\, dm$$

- Components of $\boldsymbol{\omega}$ are constants $\rightarrow$ factor out of integrals

Goal:

- Express angular momentum in terms of mass distribution

Moments and Products of Inertia

Define inertia terms:

$$I_{xx} = \int (y^2 + z^2)\, dm \quad I_{yy} = \int (z^2 + x^2)\, dm \quad I_{zz} = \int (x^2 + y^2)\, dm$$

$$I_{xy} = -\int xy\, dm \quad I_{xz} = -\int xz\, dm \quad I_{yz} = -\int yz\, dm$$

Interpretation:

- I_{xx}, I_{yy}, I_{zz} $\rightarrow$ **Moments of inertia**
- I_{xy}, I_{xz}, I_{yz} $\rightarrow$ **Products of inertia**
- Describe how mass is distributed relative to axes

Angular Momentum & Key Observations

Angular Momentum (Component Form)

$$\mathbf{H} = (I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)\mathbf{i}$$

$$+(I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z)\mathbf{j}$$

$$+(I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z)\mathbf{k}$$

Components:

$$H_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$H_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$H_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

Symmetry:

$$I_{xy} = I_{yx}, \quad I_{xz} = I_{zx}, \quad I_{yz} = I_{zy}$$

Reference & Axes

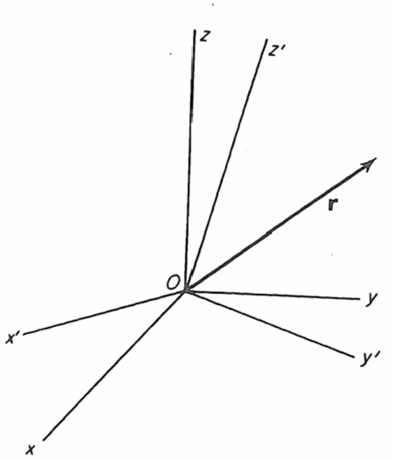
- Valid about:
  - Mass center G
  - Fixed point O
- Body-fixed axes:
  - Inertia terms constant
- Non-body-fixed axes:
  - Inertia varies with time
  - More complex analysis

Physical Insights

- Symmetry axis:
  - Simplifies inertia
- $\mathbf{H}$ depends on: $\boldsymbol{\omega}$ and Mass distribution

Setup:

- Two coordinate systems:
  - Unprimed: x-y-z
  - Primed: x'-y'-z'
- Both share the same origin O
- Axes are rotated relative to each other



Position Vector:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Goal:

- Express vectors in one system using the other

Transformation of Unit Vectors

Key Idea:

- Each unit vector can be expressed in the primed system
- Components = cosines of angles between axes

Transformation Relations:

$$\mathbf{i} = l_{x'x}\mathbf{i}' + l_{y'x}\mathbf{j}' + l_{z'x}\mathbf{k}'$$

$$\mathbf{j} = l_{x'y}\mathbf{i}' + l_{y'y}\mathbf{j}' + l_{z'y}\mathbf{k}'$$

$$\mathbf{k} = l_{x'z}\mathbf{i}' + l_{y'z}\mathbf{j}' + l_{z'z}\mathbf{k}'$$

Direction Cosines:

$$l_{ab} = \cos(\text{angle between axes } a, b)$$

Rotation Matrix Representation:

$$\begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} = \begin{bmatrix} l_{x'x} & l_{y'x} & l_{z'x} \\ l_{x'y} & l_{y'y} & l_{z'y} \\ l_{x'z} & l_{y'z} & l_{z'z} \end{bmatrix} \begin{bmatrix} \mathbf{i}' \\ \mathbf{j}' \\ \mathbf{k}' \end{bmatrix}$$

Properties:

- Orthogonal matrix
- Rows/columns are unit vectors
- Represents pure rotation

Interpretation:

- Converts coordinates between frames

Position Vector & Coordinate Transformation

Position Vector in Primed System:

$$\mathbf{r} = (l_{x'x}x + l_{x'y}y + l_{x'z}z)\mathbf{i}'$$
$$+(l_{y'x}x + l_{y'y}y + l_{y'z}z)\mathbf{j}'$$
$$+(l_{z'x}x + l_{z'y}y + l_{z'z}z)\mathbf{k}'$$

Also:

$$\mathbf{r} = x'\mathbf{i}' + y'\mathbf{j}' + z'\mathbf{k}'$$

Compare coefficients:

$$x' = l_{x'x}x + l_{x'y}y + l_{x'z}z$$
$$y' = l_{y'x}x + l_{y'y}y + l_{y'z}z$$
$$z' = l_{z'x}x + l_{z'y}y + l_{z'z}z$$

Interpretation:

- Same vector expressed in two coordinate systems
- New coordinates = weighted sum of old coordinates

Matrix Form of Coordinate Transformation

Compact Representation:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} l_{x'x} & l_{x'y} & l_{x'z} \\ l_{y'x} & l_{y'y} & l_{y'z} \\ l_{z'x} & l_{z'y} & l_{z'z} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- Matrix = **rotation matrix**
- Built from direction cosines
- Converts coordinates between frames

Abbreviated Notation:

$$\{r'\} = [l]\{r\}$$

Where:

- $\{r\}$ = coordinates in unprimed system
- $\{r'\}$ = coordinates in primed system
- $[l]$ = rotation matrix

Interpretation:

- Compact vector form of transformation
- Same relation expressed efficiently

Physical Interpretation

Two Equivalent Views:

- **Rotating Coordinate System:**
  - Vector $\mathbf{r}$ is fixed
  - Axes rotate
  - Components change $\rightarrow \{r'\}$
- **Rotating Vector:**
  - Coordinate system is fixed
  - Vector rotates in opposite direction
  - New vector $\mathbf{r}'$

Key Insight:

- Same mathematical transformation applies in both cases

Rotational Kinetic Energy in Different Coordinate Systems

Assumptions:

- Rigid body rotating with angular velocity $\boldsymbol{\omega}$
- Origin at mass center (G)
- Two coordinate systems:
  - Unprimed: x, y, z
  - Primed: x', y', z'

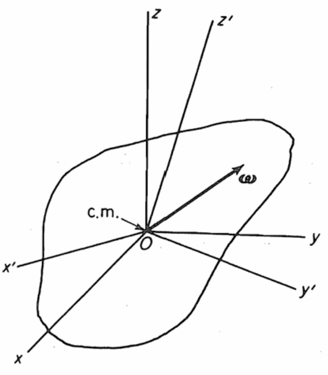
Rotational Kinetic Energy:

$$T_{\text{rot}} = \frac{1}{2}\{\omega\}^T[I]\{\omega\}$$

In primed system:

$$T_{\text{rot}} = \frac{1}{2}\{\omega'\}^T[I']\{\omega'\}$$

Visualization:



Key Insight:

- Same rigid body
- Different coordinate systems

Angular Velocity Transformation & Energy Invariance

Angular Velocity Transformation:

$$\{\omega'\} = [l]\{\omega\}$$

$$\{\omega'\}^T = \{\omega\}^T[l]^T$$

Substitute into KE:

$$T_{\text{rot}} = \frac{1}{2}\{\omega\}^T[l]^T[I'][l]\{\omega\}$$

- $[l]$ relates the two coordinate systems

Energy Invariance:

- Rotational KE is **independent of frame**
- Same $\boldsymbol{\omega}$ used in all systems

Equating expressions:

$$\frac{1}{2}\{\omega\}^T[I]\{\omega\} = \frac{1}{2}\{\omega\}^T[l]^T[I'][l]\{\omega\}$$

- Valid for **any** $\{\omega\}$
- Ensures consistency of physical laws

Transformation of Inertia Matrix

Matrix Form:

$$[I] = [l]^T [I'][l]$$

Component Form:

$$I_{ij} = \sum_{m'} \sum_{n'} l_{im'}\, l_{jn'}\, I_{m'n'}$$

Interpretation:

- Inertia matrix depends on coordinate system
- Transformation governed by rotation matrix $[l]$
- Tensor transformation (not simple vector)

Key Points:

- Each I_{ij} depends on all $I_{m'n'}$
- Produces **9 coupled equations**
- Captures full axis coupling

Important:

- $[l]$ = direction cosine matrix
- Preserves physical invariance

Inertia Matrix: Component Expansion & Properties

Example: Expanded Component (I_{xy})

$$
\begin{aligned}
I_{xy} = {} & l_{x'x}l_{x'y}I_{x'x'} + l_{y'x}l_{y'y}I_{y'y'} + l_{z'x}l_{z'y}I_{z'z'} \\
& + (l_{x'x}l_{y'y} + l_{y'x}l_{x'y})I_{x'y'} \\
& + (l_{x'x}l_{z'y} + l_{z'x}l_{x'y})I_{x'z'} \\
& + (l_{y'x}l_{z'y} + l_{z'x}l_{y'y})I_{y'z'}
\end{aligned}
$$

- Mixes direction cosines and inertia terms

Key Properties
Symmetry:

$$I_{xy} = I_{yx}, \quad I_{xz} = I_{zx}, \quad I_{yz} = I_{zy}$$

Implications:

- Inertia matrix is **symmetric**
- Only **6 independent components**

Takeaway:

- Second-order tensor
- Transformation preserves physics

Homogeneous Sphere & Orthogonality of Rotation Matrix

Homogeneous Sphere:

- Same inertia about all axes
- Origin at mass center

Result:

$$[I] = [I'] = [1]$$

- Products of inertia vanish
- Due to spherical symmetry

$$[l]^T[I'][l] = [l]^T[l] = [1]$$

Orthogonality:

$$[l]^T[l] = [1]$$

Inverse property:

$$[l][l]^{-1} = [1]$$

Thus:

$$[l]^T = [l]^{-1}$$

Conclusion:

- $[l]$ is an **orthogonal matrix**
- Rows/columns are unit and perpendicular

Inertia Transformation Forms & Key Results

Start with:

$$[I] = [l]^T [I'][l]$$

Multiply:

$$[l][I][l]^T = [l][l]^T [I'][l][l]^T$$

Using:

$$[l][l]^T = [1]$$

Result:

$$[I'] = [l][I][l]^T$$

Key Takeaways:

- $[l]^T = [l]^{-1}$
- Two equivalent forms:

$$[I] = [l]^T [I'][l]$$
$$[I'] = [l][I][l]^T$$

- Inertia matrix is a **tensor**
- Transformation preserves physics

Direction Cosines and Degrees of Freedom

Direction Cosine Constraints:

$$\sum_{m'} l_{im'}\, l_{jm'} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Compact Form:

$$\sum_{m'} l_{im'}\, l_{jm'} = \delta_{ij}$$

Where:

- δ_{ij} = **Kronecker delta**
- $\delta_{ij} = 1$ if $i = j$, else 0

Physical Meaning:

- 9 direction cosines:

$$l_{x'x},\, l_{x'y},\, l_{x'z}, \ldots$$

- 6 independent constraints:

$$\sum l_{im'} l_{jm'} = \delta_{ij}$$

Degrees of Freedom:

$$9 - 6 = 3$$

- Corresponds to **3 rotational DOF**
- Defines orientation of a rigid body

Principal Axes & Sum of Moments of Inertia

Principal Axes: Key Idea

- Study variation of inertia with rotation of axes
- Origin O is fixed (not necessarily mass center)

Important Notes:

- Inertia depends on axis orientation
- Moments of inertia are **non-negative**

Sum of Moments of Inertia

$$I_{xx} + I_{yy} + I_{zz} = 2 \int_V \rho r^2 \, dV$$

Where:

$$r^2 = x^2 + y^2 + z^2$$

- r = distance from origin
- ρdV = mass element

Insight:

- Depends only on distance
- Independent of axis orientation

Invariance and Trace of Inertia Matrix

Invariance Under Rotation

- Distance r unchanged by axis rotation
- Hence:
  $I_{xx} + I_{yy} + I_{zz} =$ constant
- Independent of coordinate system

Matrix Interpretation

- Sum of diagonal terms
- Called **trace**
  $\text{trace}([I]) = I_{xx} + I_{yy} + I_{zz}$

Trace Invariance

- Trace remains unchanged under rotation

General Property

- Trace invariant under:
  - **Orthogonal transformations**

Implication

- Physical properties do not depend on axes

Conclusion

- Rotation changes components
- But not total inertia measure

Products of Inertia and Coordinate Rotation

Effect of Rotation

- Products of inertia depend on axis orientation
- Their **signs can change**

Examples

- 180° about x-axis:
  - I_{xy}, I_{xz} change sign
  - I_{yz} unchanged
- 90° about x-axis:
  - I_{yz} changes sign

Nature of Products of Inertia

- No preferred sign
- Can be positive or negative

Implication

- Random orientation:
  - Positive/negative equally likely

Behavior

- Vary **smoothly** with rotation
- Due to smooth change in direction cosines

Principal Axes and Their Significance

Principal Axes Concept

- Always possible to find axes such that:

$$I_{xy} = I_{xz} = I_{yz} = 0$$

Inertia Matrix

$$\mathbf{I} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

- Matrix becomes **diagonal**

Definitions

- Axes $\rightarrow$ **Principal axes**
- Moments $\rightarrow$ **Principal moments**

Properties

- Axes are mutually perpendicular
- Define **principal planes**

Key Insight

- Simplifies dynamics
- Eliminates coupling between axes

Angular Momentum in Principal Axes

Assumption:

- Axes are **principal axes**
- Products of inertia = 0

Angular Momentum Components:

$$H_x = I_{xx}\omega_x$$

$$H_y = I_{yy}\omega_y$$

$$H_z = I_{zz}\omega_z$$

Key Point:

- No coupling between components

Rotation About a Principal Axis

- Example: ω along x-axis

$$\omega_y = 0, \quad \omega_z = 0$$

$$H_y = 0, \quad H_z = 0$$

Result:

- $\mathbf{H} \parallel \boldsymbol{\omega}$

$$\mathbf{H} = I\,\boldsymbol{\omega}$$

- I = moment of inertia about that axis

General Angular Momentum and Parallelism Condition

General Angular Momentum

$$H_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$H_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$H_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

Observation:

- Components are **coupled**
- $\mathbf{H}$ depends on all components of $\boldsymbol{\omega}$

Case: $\boldsymbol{\omega}$ along x-axis

$$\omega_y = 0, \quad \omega_z = 0$$

$$H_x = I_{xx}\omega_x$$

$$H_y = I_{yx}\omega_x, \quad H_z = I_{zx}\omega_x$$

Observation:

- $\mathbf{H}$ has y, z components
- $\mathbf{H} \not\parallel \boldsymbol{\omega}$ (in general)

Condition for Parallelism:

- $I_{yx} = I_{zx} = 0$

Principal Axes and Parallelism Condition

Key Conclusion:

- $\mathbf{H} \parallel \boldsymbol{\omega}$ **only if rotation is about a principal axis**

Reason:

- Products of inertia vanish:

$$I_{xy} = I_{xz} = I_{yz} = 0$$

Final Statement:

- For a rigid body about point O:

$$\mathbf{H} \parallel \boldsymbol{\omega} \quad \text{iff} \quad \text{rotation is about a principal axis}$$

Angular Momentum and Eigenvalue Formulation

Matrix Form of Angular Momentum

$$\begin{Bmatrix} H_x \\ H_y \\ H_z \end{Bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

If $\mathbf{H} \parallel \omega$

$$\begin{Bmatrix} H_x \\ H_y \\ H_z \end{Bmatrix} = I \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

- I = scalar moment of inertia
- Rotation about **principal axis**

Eigenvalue Equation

$$\left(\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} - I\mathbf{I} \right) \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = 0$$

Non-trivial solution:

$$\det \begin{bmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{bmatrix} = 0$$

- Determines principal moments

Finding Principal Directions

Key Idea

- Substitute each principal moment I into:

$$(\mathbf{I} - I\mathbf{I})\boldsymbol{\omega} = 0$$

Important Observation

- Only **ratios** of components:

$$\omega_x, \ \omega_y, \ \omega_z$$

- Absolute magnitude cannot be determined

Reason

- Scaling $\boldsymbol{\omega}$ does not affect equation

Solving for Direction Ratios

- Use two independent equations
- Assume $\omega_x \neq 0$

Divide by ω_x:

$$(I_{yy} - I)\frac{\omega_y}{\omega_x} + I_{yz}\frac{\omega_z}{\omega_x} = -I_{yx}$$

$$I_{zy}\frac{\omega_y}{\omega_x} + (I_{zz} - I)\frac{\omega_z}{\omega_x} = -I_{zx}$$

Result

- Solve for:

$$\frac{\omega_y}{\omega_x}, \ \frac{\omega_z}{\omega_x}$$

Principal Axis Direction from Direction Ratios

Matrix Form (Compact Solution):

$$\begin{Bmatrix} \frac{\omega_y}{\omega_x} \\ \frac{\omega_z}{\omega_x} \end{Bmatrix} = \begin{bmatrix} (I_{yy} - I) & I_{yz} \\ I_{zy} & (I_{zz} - I) \end{bmatrix}^{-1} \begin{Bmatrix} -I_{yx} \\ -I_{zx} \end{Bmatrix}$$

- Gives direction ratios for each I
- Each solution $\Rightarrow$ one principal axis

Final Step:

- Normalize to obtain unit vector

Physical Interpretation:

- Ratios define direction of axis
- Example:
  - Set $\omega_x = 1$
  - Solve for $\omega_y,\ \omega_z$
- Direction of $\boldsymbol{\omega} \Rightarrow$ principal axis direction

Important Notes:

- Any component can be reference
- If chosen component $= 0$:
  - Choose another non-zero component
- Verify with full equations

Eigenvalue Formulation of Principal Axes

Core Equation:

$$[I]\{\omega\} - I\{\omega\} = 0$$

Interpretation:

- Standard **eigenvalue problem**
- $I \rightarrow$ **eigenvalues** (principal moments)
- $\{\omega\} \rightarrow$ **eigenvectors** (principal directions)

Key Insight:

- Solve:

$$([I] - I\mathbf{I})\{\omega\} = 0$$

- Non-trivial solution requires:

$$\det([I] - I\mathbf{I}) = 0$$

Result:

- 3 eigenvalues $\Rightarrow$ principal moments
- 3 eigenvectors $\Rightarrow$ principal axes

Principal Directions & Interpretation

Steps (Eigenvalue Method):

Result:

- Eigenvectors $\rightarrow$ principal axes
- Eigenvalues $\rightarrow$ principal moments

In this orientation:

$$\mathbf{I} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

- Matrix is **diagonalized**
- Axes are **principal axes**

$$\mathbf{H} = I_{xx}\,\omega_x\,\mathbf{n}_1 + I_{yy}\,\omega_y\,\mathbf{n}_2 + I_{zz}\,\omega_z\,\mathbf{n}_3$$

- I_{xx}, I_{yy}, I_{zz} are: **Principal moments of inertia**

- Represent:
    - Maximum moment
    - Minimum moment
    - Intermediate value

Important:

- Principal axes always exist
- Simplify angular momentum relations

Examples:

- Sphere $\rightarrow$ all axes principal
- Rectangular block $\rightarrow$ edges are principal axes

Example Problem: Principal Moments of Inertia

Given:

- A rigid body with inertia matrix (about mass center G):

$$[I] = \begin{bmatrix} 150 & 0 & -100 \\ 0 & 250 & 0 \\ -100 & 0 & 300 \end{bmatrix} \quad (\text{lb}\cdot\text{sec}^2/\text{ft})$$

Required:

- Determine the **principal moments of inertia**
- Find the **principal axes (directions)**
- Obtain a **coordinate transformation** that diagonalizes $[I]$

Notes:

- Reference point is at the **center of mass**
- Solve using the **eigenvalue problem**:

$$\det([I] - I\mathbf{I}) = 0$$

Solution: Principal Moments of Inertia

Step 1: Characteristic Equation

$$\det([I] - I\mathbf{I}) = \begin{vmatrix} 150 - I & 0 & -100 \\ 0 & 250 - I & 0 \\ -100 & 0 & 300 - I \end{vmatrix} = 0$$

Step 2: Expand Determinant

$$(250 - I)\Big[(150 - I)(300 - I) - 10^4\Big] = 0$$

$$(250 - I)(I^2 - 450I + 3.5 \times 10^4) = 0$$

Step 3: Solve for Eigenvalues

$$I_1 = 100, \quad I_2 = 250, \quad I_3 = 350 \quad (\text{lb·sec}^2/\text{ft})$$

Step 4: Direction Equations

$$(150 - I)\omega_x - 100\omega_z = 0$$

Principal Axis Directions (Eigenvectors)

Case 1: $I_1 = 100$

- From equations:

$$(250 - I)\omega_y = 0 \Rightarrow \omega_y = 0$$

- Solve:

$$\frac{\omega_z}{\omega_x} = \frac{1}{2}$$

- Direction lies in xz-**plane**

- Angle with x-axis:

$$\alpha_1 = \tan^{-1}\left(\frac{1}{2}\right)$$

Case 2: $I_3 = 350$

- Again: $\omega_y = 0$

- Solve:

$$\frac{\omega_z}{\omega_x} = -2$$

- Direction lies in xz-**plane**

Third Principal Axis and Geometric Insight

Case 3: $I_2 = 250$

- From:

$$(250 - I)\omega_y = 0 \Rightarrow \omega_y \neq 0$$

- Remaining equations $\Rightarrow \omega_x = \omega_z = 0$
- Axis lies along:

$$y\textbf{-axis}$$

Geometric Interpretation:

- Two principal axes lie in the xz-**plane**
- Third axis is **perpendicular** $\rightarrow$ along y

Important Note:

- Avoid dividing by a component that may be zero
- Always verify using original equations

Visualization of Principal Axes Rotation

Equivalent Mass System:

- Four particles (unit mass) located at:

$$(0, \sqrt{50}, 0),\ (0, -\sqrt{50}, 0),\ (10, 0, 5),\ (-10, 0, -5)$$

Interpretation:

- Principal axes obtained from direction ratios
- Primed system chosen to:
  - Form a right-handed coordinate system
  - Require minimum rotation from original axes

Rotation Details:

- Rotation occurs about y-**axis**
- Angle:

$$\alpha_1 = 26.6^\circ$$

Direction Cosine Matrix and Diagonalization

Direction Cosine Matrix:

$$[l] = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{bmatrix}$$

Verification:

$$[I'] = [l][I][l]^T$$

$$[I'] = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 250 & 0 \\ 0 & 0 & 350 \end{bmatrix}$$

Conclusion:

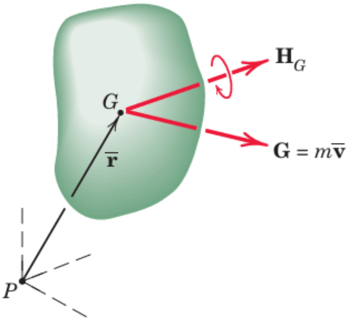
- Inertia matrix is **diagonalized**
- Confirms principal moments:

$$I_1 = 100, \quad I_2 = 250, \quad I_3 = 350$$

Transfer Principle for Angular Momentum

Concept:

- Rigid body momentum can be represented by:
  - Linear momentum: $\mathbf{G} = m\bar{\mathbf{v}}$
  - Angular momentum about G: $\mathbf{H}_G$
- Angular momentum about any point P:
  - Sum of:
    - Angular momentum about G
    - Moment of $\mathbf{G}$ about P



Transfer Equation:

$$\boxed{\mathbf{H}_P = \mathbf{H}_G + \mathbf{r} \times \mathbf{G}}$$

- $\mathbf{r}$: position vector from P to G

Kinetic Energy of a Rigid Body

General Expression:

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\sum m_i \left|\dot{\boldsymbol{\rho}}_i\right|^2$$

Interpretation:

- $\bar{v}$: velocity of mass center
- $\boldsymbol{\rho}_i$: relative position vector

Energy Decomposition:

- $\frac{1}{2}m\bar{v}^2 \rightarrow$ **translation**
- $\frac{1}{2}\sum m_i|\dot{\boldsymbol{\rho}}_i|^2 \rightarrow$ **relative motion**

Translational Kinetic Energy:

$$\frac{1}{2}m\bar{v}^2 = \frac{1}{2}m\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \frac{1}{2}\bar{\mathbf{v}} \cdot \mathbf{G}$$

- $\dot{\mathbf{r}} = \bar{\mathbf{v}}$
- $\mathbf{G} = m\bar{\mathbf{v}}$

Key Insight:

- Depends only on mass center motion

Rotational Kinetic Energy of a Rigid Body

Rigid Body Kinematics:

$$\dot{\boldsymbol{\rho}}_i = \boldsymbol{\omega} \times \boldsymbol{\rho}_i$$

Substitute into KE:

$$\frac{1}{2}\sum m_i|\dot{\boldsymbol{\rho}}_i|^2 = \frac{1}{2}\sum m_i(\boldsymbol{\omega}\times\boldsymbol{\rho}_i)\cdot(\boldsymbol{\omega}\times\boldsymbol{\rho}_i)$$

Use identity:

$$\mathbf{P}\cdot(\mathbf{Q}\times\mathbf{R}) = (\mathbf{P}\times\mathbf{Q})\cdot\mathbf{R}$$

$$(\boldsymbol{\omega}\times\boldsymbol{\rho}_i)\cdot(\boldsymbol{\omega}\times\boldsymbol{\rho}_i) = \boldsymbol{\omega}\cdot[\boldsymbol{\rho}_i\times(\boldsymbol{\omega}\times\boldsymbol{\rho}_i)]$$

Factor out ω:

$$\frac{1}{2}\sum m_i|\dot{\boldsymbol{\rho}}_i|^2 = \frac{1}{2}\boldsymbol{\omega}\cdot\sum \boldsymbol{\rho}_i\times m_i(\boldsymbol{\omega}\times\boldsymbol{\rho}_i)$$

Recognize angular momentum:

$$\mathbf{H}_G = \sum \boldsymbol{\rho}_i \times m_i\mathbf{v}_i$$

Final Result:

$$\boxed{\frac{1}{2}\sum m_i|\dot{\boldsymbol{\rho}}_i|^2 = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{H}_G}$$

Interpretation:

- Rotational KE about mass center

Kinetic Energy of a Rigid Body (Continuation)

General Result:

$$\boxed{T = \frac{1}{2}\bar{\mathbf{v}} \cdot \mathbf{G} + \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{H}_G}$$

Expanded Form:

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\left(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 - I_{xy}\omega_x\omega_y - I_{xz}\omega_x\omega_z - I_{yz}\omega_y\omega_z\right)$$

Principal Axes Simplification:

$$\boxed{T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\left(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2\right)}$$

Special Case (Rotation about fixed point O):

$$\boxed{T = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{H}_O}$$

- $\mathbf{H}_O$: angular momentum about point O

Momentum and Energy Equations of Motion

Overview:

- Based on:
  - Angular momentum
  - Inertia properties
  - Kinetic energy
- Used to derive:
  - Momentum equations
  - Energy equations

Momentum Equations:

$$\boxed{\sum \mathbf{F} = \dot{\mathbf{G}}, \quad \sum \mathbf{M} = \dot{\mathbf{H}}}$$

- Valid for constant mass systems

Moment Equation in Moving Axes

General form:

$$\sum \mathbf{M} = \dot{\mathbf{H}}$$

Rotating coordinates (x-y-z):

$$\sum \mathbf{M} = \left(\frac{d\mathbf{H}}{dt}\right)_{xyz} + \boldsymbol{\Omega} \times \mathbf{H}$$

Interpretation:

- Change in magnitude
- Change in direction

Moment Equations in Component Form

General (Rotating Axes):

$$\sum \mathbf{M} = (\dot{H}_x - H_y\Omega_z + H_z\Omega_y)\mathbf{i}$$

$$+(\dot{H}_y - H_z\Omega_x + H_x\Omega_z)\mathbf{j}$$

$$+(\dot{H}_z - H_x\Omega_y + H_y\Omega_x)\mathbf{k}$$

Applicability:

- Valid about:
  - Fixed point O
  - Mass center G

Body-Fixed Axes:

$$\boldsymbol{\Omega} = \boldsymbol{\omega}$$

Simplified Equations:

$$\sum M_x = \dot{H}_x - H_y\omega_z + H_z\omega_y$$

$$\sum M_y = \dot{H}_y - H_z\omega_x + H_x\omega_z$$

$$\sum M_z = \dot{H}_z - H_x\omega_y + H_y\omega_x$$

Key Insight:

- Inertia terms constant
- Most practical form

Euler's Equations of Motion

Starting Point:

$$\sum \mathbf{M} = \dot{\mathbf{H}}$$

In rotating (body-fixed) axes:

$$\sum \mathbf{M} = \left(\frac{d\mathbf{H}}{dt}\right)_{xyz} + \boldsymbol{\omega} \times \mathbf{H}$$

Angular momentum (principal axes):

$$\mathbf{H} = I_{xx}\omega_x \mathbf{i} + I_{yy}\omega_y \mathbf{j} + I_{zz}\omega_z \mathbf{k}$$

Time derivative:

$$\dot{\mathbf{H}} = I_{xx}\dot{\omega}_x \mathbf{i} + I_{yy}\dot{\omega}_y \mathbf{j} + I_{zz}\dot{\omega}_z \mathbf{k}$$

Compute cross product:

$$\boldsymbol{\omega} \times \mathbf{H} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \begin{bmatrix} I_{xx}\omega_x \\ I_{yy}\omega_y \\ I_{zz}\omega_z \end{bmatrix}$$

Substitute into moment equation:

$$\boxed{\begin{aligned} \sum M_x &= I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y\omega_z \\ \sum M_y &= I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\omega_x \\ \sum M_z &= I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y \end{aligned}}$$

These are Euler's Equations

Parallel-Plane Motion

Definition:

- All particles move in planes parallel to a fixed plane
- Special case of 3D rigid-body motion

Key Characteristics:

- Lines normal to the plane remain parallel
- Motion analyzed in a single plane

Reference Frame:

- Origin at mass center G
- Axes x-y lie in plane of motion

Angular Velocity:

Angular Momentum Components:

$$H_x = -I_{xz}\omega_z, \quad H_y = -I_{yz}\omega_z$$

$$H_z = I_{zz}\omega_z$$

Moment Equations:

$$\boxed{\begin{aligned} \sum M_x &= -I_{xz}\dot{\omega}_z + I_{yz}\omega_z^2 \\ \sum M_y &= -I_{yz}\dot{\omega}_z - I_{xz}\omega_z^2 \\ \sum M_z &= I_{zz}\dot{\omega}_z \end{aligned}}$$

Equations of Motion (Plane Motion)

Force Equations:

$$\sum F_x = ma_x, \quad \sum F_y = ma_y, \quad \sum F_z = 0$$

Important Observations:

- Motion is confined to a plane $\rightarrow$ no z translation
- Rotation occurs only about z-axis
- Third moment equation reduces to planar rotation:

$$\sum M_z = I_{zz}\dot{\omega}_z$$

Applications:

- Rotating machinery (imbalance analysis)
- Rolling bodies (wheels, disks)

Parallel-Plane Motion

Definition:

- All particles move in planes parallel to a fixed plane
- Special case of 3D rigid-body motion

Key Characteristics:

- Lines normal to the plane remain parallel
- Motion analyzed in a single plane

Reference Frame:

- Origin at mass center G
- Axes x-y lie in plane of motion

Angular Velocity:

$$\omega_x = 0, \quad \omega_y = 0, \quad \omega_z \neq 0$$

Angular Momentum Components:

$$H_x = -I_{xz}\omega_z, \quad H_y = -I_{yz}\omega_z$$

$$H_z = I_{zz}\omega_z$$

Moment Equations:

$$\boxed{\begin{aligned} \sum M_x &= -I_{xz}\dot{\omega}_z + I_{yz}\omega_z^2 \\ \sum M_y &= -I_{yz}\dot{\omega}_z - I_{xz}\omega_z^2 \\ \sum M_z &= I_{zz}\dot{\omega}_z \end{aligned}}$$