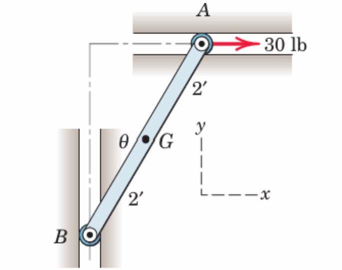
Kinetics of Rigid Bodies

Sample Problem 6/7: Slender Bar in Constrained Motion



- Slender bar AB:
  - Weight $= 60$ lb
  - Length $= 4$ ft $(2' + 2')$
- Motion:
  - Moves in vertical plane
  - Ends constrained in:
- Horizontal guide at A
- Vertical guide at B
- Applied force:

$$F = 30 \text{ lb at } A$$

- Initial condition:

$$\theta = 30^\circ, \quad \text{released from rest}$$

- **Determine:**
  - Angular acceleration α of the bar
  - Forces on rollers at A and B

Kinematics: Acceleration of Mass Center G

- Constrained motion $\Rightarrow$ relate $\mathbf{a}_G$ and $\boldsymbol{\alpha}$
- Relative acceleration:

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$$

- Since A moves horizontally and B vertically:
  - $\mathbf{a}_A \rightarrow$ horizontal
  - $\mathbf{a}_B \rightarrow$ vertical
- Relative term:

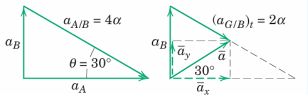
$$\mathbf{a}_{A/B} = \boldsymbol{\alpha} \times \mathbf{r}_{A/B} \quad \Rightarrow \quad |\mathbf{a}_{A/B}| = \alpha(4 \text{ ft})$$

- Acceleration polygon gives:

$$\mathbf{a}_{A/B} = 4\alpha$$

- Now for mass center:

$$\mathbf{a}_G = \mathbf{a}_B + \mathbf{a}_{G/B}$$



- Relative acceleration:

$$\mathbf{a}_{G/B} = \alpha(2 \text{ ft})$$

- From geometry ($\theta = 30^\circ$):

$$\bar{a} = 2\alpha$$

- Resolve components:

$$a_x = \bar{a}\cos 30^\circ = 2\alpha\cos 30^\circ = 1.732\alpha$$

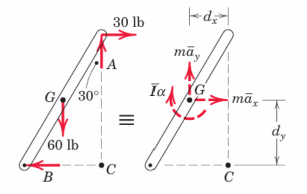
$$a_y = \bar{a}\sin 30^\circ = 2\alpha\sin 30^\circ = 1.0\alpha$$

- Equations of motion:

$$\sum M_G: \quad 30(2\cos 30^\circ) - A(2\sin 30^\circ) + B(2\cos 30^\circ) = \frac{1}{12}\frac{60}{32.2}(4^2)\alpha$$

$$\sum F_x: \quad 30 - B = \frac{60}{32.2}(1.732\alpha)$$

$$\sum F_y: \quad A - 60 = \frac{60}{32.2}(1.0\alpha)$$



- Solution:

$$A = 68.2\text{ lb}, \quad B = 15.74\text{ lb}, \quad \alpha = 4.42\text{ rad/s}^2$$

Solution: Alternative Moment Approach

- Use moment about point C:

$$\sum M_C = \bar{I}\alpha + \sum m\bar{a}d$$

- Substitution:

$$30(4\cos 30^\circ) - 60(2\sin 30^\circ) = \frac{1}{12}\frac{60}{32.2}(4^2)\alpha$$

$$+\frac{60}{32.2}(1.732\alpha)(2\cos 30^\circ) + \frac{60}{32.2}(1.0\alpha)(2\sin 30^\circ)$$

- Solve:

$$43.9 = 9.94\alpha \Rightarrow \alpha = 4.42 \text{ rad/s}^2$$

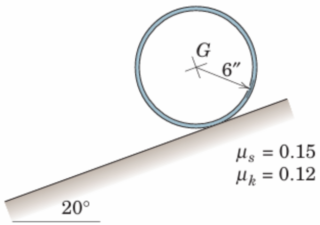
- Back-substitute:

$$A - 60 = \frac{60}{32.2}(1.0)(4.42) \Rightarrow A = 68.2 \text{ lb}$$

$$30 - B = \frac{60}{32.2}(1.732)(4.42) \Rightarrow B = 15.74 \text{ lb}$$

- Key insight:
  - Using point C eliminates A and B
  - Gives α directly
- Strategy:
  - Choose moment center wisely
  - Reduces number of unknowns
- Final answers:
  - $\alpha = 4.42$ rad/s^2
  - $A = 68.2$ lb
  - $B = 15.74$ lb

Sample Problem 6/5: Rolling Hoop on Incline



- A metal hoop of radius:

$$r = 6 \text{ in.}$$

- Released from rest on a:

$$20^\circ \text{ incline}$$

- Coefficients of friction:

$$\mu_s = 0.15, \quad \mu_k = 0.12$$

- **Determine:**
  - Angular acceleration α of the hoop
  - Time t to move:

$$10 \text{ ft down the incline}$$

Solution: Dynamics and Rolling Assumption

- Forces: mg, N, friction F
- Assume **pure rolling**: $a = r\alpha$
- Equations of motion:

$$\sum F_x : \quad mg \sin 20^\circ - F = m\bar{a}$$

$$\sum F_y : \quad N - mg \cos 20^\circ = 0$$

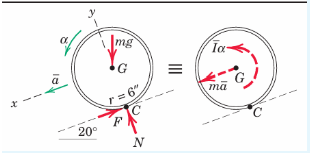
$$\sum M_G : \quad Fr = mr^2\alpha$$

- Using $a = r\alpha$:

$$\bar{a} = \frac{g}{2} \sin 20^\circ = 5.51 \text{ ft/s}^2$$

- Check friction:

$$F = 0.1710mg, \quad N = 0.940mg$$



$$F_{\max} = \mu_s N = 0.1410mg$$

- Since $F > F_{\max}$:
  - **Rolling assumption invalid**
  - Hoop **slips**

Solution: Sliding Motion and Final Results

- Use **kinetic friction**:

$$F = \mu_k N = 0.12(0.940mg) = 0.1128mg$$

- Force equation:

$$mg \sin 20^\circ - 0.1128mg = m\bar{a}$$

$$\bar{a} = 0.229(32.2) = 7.38 \text{ ft/s}^2$$

- Moment equation:

$$0.1128mg(r) = mr^2\alpha$$

$$\alpha = 7.26 \text{ rad/s}^2$$

- Time to travel 10 ft:

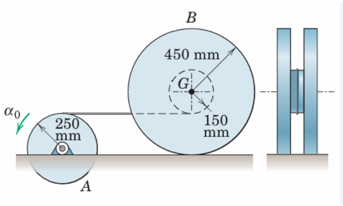
$$x = \frac{1}{2}at^2$$

$$x = \frac{1}{2}at^2$$

$$t = \sqrt{\frac{2x}{a}} = \sqrt{\frac{20}{7.38}} = 1.646 \text{ s}$$

- **Final Answers:**
  - $\alpha = 7.26$ rad/s^2
  - $t = 1.646$ s
- Key takeaway:
  - Always verify rolling condition
  - Compare F with $\mu_s N$
- If violated:
  - Switch to kinetic friction
  - Re-solve equations

Sample Problem 6/6: Rolling Spool Driven by Cable



- Drum A has constant angular acceleration:

$$\alpha_0 = 3 \text{ rad/s}^2$$

- Drives a **70-kg spool** B via cable
- Cable wraps around the **inner hub**

- Geometry:
  - Outer radius $= 450$ mm
  - Inner hub radius $= 150$ mm
  - Radius of gyration:

$$k_G = 250 \text{ mm}$$

- Surface condition:

$$\mu_s = 0.25$$

- **Determine:**
  - Tension T in the cable
  - Friction force F on the spool

Solution: Kinematics and Equations of Motion

- Cable acceleration:

$$a = r\alpha_0 = 0.25(3) = 0.75 \text{ m/s}^2$$

- Assume **no slipping**:

$$\alpha = \frac{a}{r} = \frac{0.75}{0.30} = 2.5 \text{ rad/s}^2$$

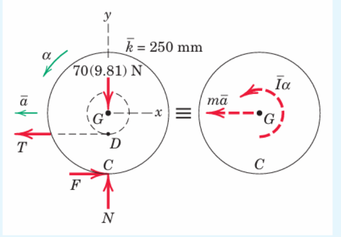
- Mass-center acceleration:

$$\bar{a} = r\alpha = 0.45(2.5) = 1.125 \text{ m/s}^2$$

- Equations of motion:

$$\sum F_x : \quad F - T = 70(-1.125)$$

$$\sum F_y : \quad N - 70(9.81) = 0 \Rightarrow N = 687 \text{ N}$$



$$\sum M_G : \quad F(0.450) - T(0.150) = 70(0.25)^2(2.5)$$

- Solve:

$$F = 75.8 \text{ N}, \quad T = 154.6 \text{ N}$$

Solution: Friction Check and Alternative Method

- Check rolling condition:

$$F_{\max} = \mu_s N = 0.25(687) = 171.7 \text{ N}$$

- Since:

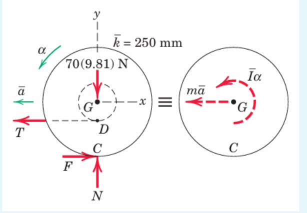
$$F = 75.8 \text{ N} < F_{\max}$$

  - Assumption of **no slipping is valid**

- If $\mu_s = 0.1$:

$$F_{\max} = 68.7 \text{ N} < 75.8 \text{ N}$$

  - Spool would **slip**
  - Then $a \neq r\alpha$



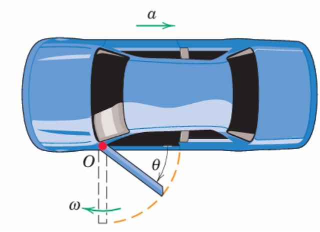
- Alternative (moment about C):

$$\sum M_C = I\alpha + m\bar{a}r$$

$$0.3T = 70(0.25)^2(2.5) + 70(1.1$$

$$T = 154.6 \text{ N}$$

Sample Problem 6/8: Car Door Dynamics



- A car undergoes a constant rearward acceleration: a
- The door is slightly open and free to rotate about hinge O
- As the car accelerates, the door swings with angle θ

- Given:
  - Mass of door: m
  - Distance of mass center from hinge: r
  - Radius of gyration about O: k_O
- **Determine:**
  - Angular velocity ω when door passes $90°$
  - Hinge reaction components at O for any θ

Solution: Kinematics and Acceleration Components

- Angular velocity ω varies with $\theta \Rightarrow$ need $\alpha(\theta)$
- Strategy:
  - Use moment equation about hinge O
  - Then integrate to obtain ω
- Free-body diagram:
  - Only hinge reaction components (x, y directions)
- Kinetic diagram:
  - Resultant force $m\bar{\mathbf{a}}$
  - Resultant couple $I_O\alpha$
- Relative acceleration:

$$\bar{\mathbf{a}} = \mathbf{a}_G = \mathbf{a}_O + (\mathbf{a}_{G/O})_n + (\mathbf{a}_{G/O})_t$$

- Components of acceleration:

$$ma_O = ma$$

$$m(a_{G/O})_n = mr\omega^2$$

$$m(a_{G/O})_t = mr\alpha$$

- Definitions:

$$\omega = \dot{\theta}, \quad \alpha = \ddot{\theta}$$

- Key idea:
  - Acceleration of G has:
    - Base motion (car acceleration)
    - Normal component

Solution: Moment Equation and Angular Velocity

- Point O has acceleration has a
- with O as moment center:

$$\sum M_O = I_O\alpha + \bar{\boldsymbol{\rho}} \times m\mathbf{a}_O$$

- Scalar form:

$$0 = I_O\alpha - ma\bar{r}\sin\theta$$

- Let $\rho = r_{G/O} = \bar{r}$ and $I_O = mk_O^2$:

$$0 = mk_O^2\alpha - ma\bar{r}\sin\theta$$

- Solve: $\boxed{\alpha = \frac{a\bar{r}}{k_O^2}\sin\theta}$

- Integrate to obtain ω:

$$\omega\,d\omega = \alpha\,d\theta$$

$$\int_0^\omega \omega\,d\omega = \int_0^\theta \frac{a\bar{r}}{k_O^2}\sin\theta\,d\theta$$

$$\omega^2 = \frac{2a\bar{r}}{k_O^2}(1-\cos\theta)$$

- At $\theta = \frac{\pi}{2}$:

$$\boxed{\omega = \frac{1}{k_O}\sqrt{2a\bar{r}}}$$

Solution: Hinge Reaction Components O_x, O_y

- Force equations:

$$\sum F_x = m\bar{a}_x$$
$$O_x = ma - m\bar{r}\omega^2 \cos\theta - m\bar{r}\alpha \sin\theta$$

- Substitute:

$$\omega^2 = \frac{2a\bar{r}}{k_O^2}(1-\cos\theta),\ \alpha = \frac{a\bar{r}}{k_O^2}\sin\theta$$
$$O_x = m\left[a - \frac{2a\bar{r}^2}{k_O^2}(1-\cos\theta)\cos\theta - \frac{a\bar{r}^2}{k_O^2}\sin^2\theta\right]$$

$$\boxed{O_x = ma\left[1 - \frac{\bar{r}^2}{k_O^2}(1 + 2\cos\theta - 3\cos^2\theta)\right]}$$

- Force equation:

$$\sum F_y = m\bar{a}_y$$

$$O_y = m\bar{r}\alpha\cos\theta - m\bar{r}\omega^2\sin\theta$$

- Substitute:

$$O_y = m\bar{r}\frac{a\bar{r}}{k_O^2}\sin\theta\cos\theta - m\bar{r}\frac{2a\bar{r}}{k_O^2}(1-\cos\theta)\sin\theta$$

$$\boxed{O_y = \frac{ma\bar{r}^2}{k_O^2}(3\cos\theta - 2)\sin\theta}$$