$$\max_t |h_i(\rho, l, t)| < C(l)\exp(-\nu\rho).$$

Hence problems (16) can be solved uniquely and their solutions satisfy the inequalities

$$\max_t |\Pi_i y| < C(l)\exp(-\nu\rho), \qquad \rho\to\infty, \qquad i=0,1,\ldots .$$

5. To estimate the remaining term we introduce the notation $y-U_n=\xi_n$, where U_n is the partial sum of series (13). For ξ_n we obtain

$$\mu^2\Delta_r\xi_n = \frac{\partial}{\partial t}\xi_n + a\xi_n + \mu[F(r,t,U_n+\xi_n) - F(r,t,U_n)] + \left[f(r,t) + \mu F(r,t,U_n) - \mu^2\Delta_r U_n + \frac{\partial}{\partial t}U_n + aU_n\right],$$

$$\xi_n(r,t)|_{\partial\Omega}=0, \qquad \xi_n(r,t)=\xi_n(r,t+2\pi).$$

Considering (15) and (16), and also that

$$F(r,t,U_n) = \sum_{i=0}^{n}\mu^i F_n + \sum_{i=0}^{n}\mu^i \Pi_n F + O(\mu^{n+1}),$$

$$F(r,t,U_n+\xi_n) - F(r,t,U_n) = \xi_n \int_0^1 \frac{\partial}{\partial y} F(r,t,U_n+\theta\xi_n)\,d\theta,$$

we have

$$\mu^2\Delta_r\xi_n = \frac{\partial}{\partial t}\xi_n + a\xi_n + \xi_n \int_0^1 \frac{\partial}{\partial y}F(r,t,U_n+\theta\xi_n)\,d\theta + O(\mu^{n+1}),$$

whence, according to Theorem 2, it is easy to obtain the required estimate: $\|\xi_n\|=O(\mu^{n+1})$ for fairly small μ.

Thus the following theorem is proved:

Theorem 3. If the functions $f(r,t)$ and $F(r,t,y)$ satisfy conditions (12), then when $\mu<a/L$ a unique 2π-periodic solution of problem (1), (2) exists and series /13/ is an asymptotic expansion of this solution with respect to the parameter μ.

Remark. The results obained also hold for the set of equations

$$\mu^2\Delta_r z = \frac{\partial}{\partial t}z + Az + \mu F(r,t,z) + f(r,t),$$

where z, f, F are vectors and A is a matrix with eigen values which have $\mathrm{Re}\lambda_i>0$.

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ON A LAMINAR MIXING LAYER AT THE BOUNDARY BETWEEN TWO FLOWS*

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The distribution of fluid velocities in a laminar mixing layer at the boundary between two flows is described by a system of Prandtl equations with certain boundary conditions. The existence and uniqueness theorems for solving a boundary value problem which describes the laminar mixing layer are established.

1. A stationary layer of the mixing of two flows is described by the system of equations

$$uu_x + vu_y = \nu u_{yy} - p_x, \qquad u_x + v_y = 0 \tag{1}$$

in the domain $D=\{0<x<X, -\infty<y<+\infty\}$ with the conditions

$$u(0,y)=u_0(y), \qquad v(x,0)=0, \qquad u(x,y)\to U_1(x) \text{ as } y\to\infty, \tag{2a}$$

$$u(x,y)\to U_2(x) \text{ as } y\to-\infty. \tag{2b}$$

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Here $u(x,y)$ and $v(x,y)$ are the components of the fluid velocity, $u_0(y)$, $U_1(x)>0$, $U_2(x)>0$ are known functions, and $\nu>0$. The Bernoulli equation is $U_i^2(x)+2p(x)=\text{const}$, $i=1,2$. In boundary-layer theory, the pressure across the mixing layer does not vary and therefore $U_1^2(x)-U_2^2(x)=\text{const}$. The condition $v(x,0)=0$ is obtained assuming that $y=0$ is a dividing streamline.

We shall assume that $u_0(y)>0$, $u_0(y)\to U_1(0)$ as $y\to\infty$, and $u_0(y)\to U_2(0)$ as $y\to-\infty$; the function $p_x(x)$ is continuously differentiable, and $u_0(y)$, $u_0'(y)$, $u_0''(y)$ are bounded for $-\infty<y<+\infty$, and satisfy Hölder's condition.

The following theorem expresses the main finding of the present paper.

Theorem 1. Under the above assumptions, for a certain X there exists in domain D a solution u, v of problem (1), (2), which possesses the following properties: $u(x,y)$ is continuous and bounded in $\bar{D}$, $u>0$, u_y and u_{yy} are continuous and bounded in D, and u_x and v_y are continuous and bounded on any compact set in $\bar{D}$. If $|u_0'(y)|\leqslant k_1\exp(-k_2|y|)$ as $|y|\to\infty$, and k_1, $k_2>0$ are constants, then u_x and v_y are bounded in D. If $dp/dx\leqslant 0$, then such a solution exists in domain D for any finite X. The solution of problem (1), (2), having the above properties, is unique.

Certain auxiliary results discussed below are necessary to prove Theorem 1.

2. Following /1/, we introduce the independent variables

$$x=x,\quad \psi=\psi(x,y)$$

such that $u=\partial\psi/\partial y$, $v=-\partial\psi/\partial x$, $\psi(x,0)=0$, and also a new unknown function

$$w(x,\psi)=u^2.$$

Therefore

$$y=\int_0^{\psi}\frac{ds}{w^{1/2}(x,s)}.$$

As a result, system (1) with conditions (2) is reduced to the equation

$$L(w)\equiv\nu w^{1/2}\partial^2w/\partial\psi^2-\partial w/\partial x=2dp/dx \text{ in the region } \Omega=\{0<x<X,\ -\infty<\psi<+\infty\} \tag{3}$$

with the conditions

$$w(0,\psi)=w_0(\psi),\quad w(x,\psi)\to U_1^2(x),\quad \psi\to\infty,\qquad w(x,\psi)\to U_2^2(x),\quad \psi\to-\infty, \tag{4}$$

where

$$w_0\left(\int_0^{y}u_0(s)\,ds\right)=u_0^2(y).$$

We shall prove the existence and uniqueness of problem (3), (4) on which Theorem 1 is based.

Consider Eq.(3) in the domain $D_N=\{0<x<X,\ -N<\psi<N\}$, $N>0$ with the conditions

$$w(0,\psi)=w_0(\psi),\qquad w(x,N)=w_0(N)\exp[\mu(N)x/w_0(N)], \tag{5a}$$

$$w(x,-N)=w_0(-N)\exp[\mu(-N)x/w_0(-N)], \tag{5b}$$

where $\mu(\psi)=\nu w_0^{1/2}(\psi)w_0''(\psi)-2p'(0)$.

Let us show that as $N\to\infty$, for a certain X the solutions of problem (3), (5) converge to the solution of (3), (4).

Lemma 1. There exists X such that in the region D_N for the solution $w_N(x,\psi)$ of problem (3), (5), the following inequality holds:

$$w_N(x,\psi)\geqslant C\exp(-\alpha x), \tag{6}$$

where the constants C and α are positive and do not depend on N. If $dp/dx\leqslant 0$, then the inequality of the form (6) is satisfied for any finite X.

Proof. Consider the function

$$\Phi(x,\psi)=C\exp(-\alpha x),\qquad \alpha>0,\quad 0<C\leqslant\min_{\psi} w_0(\psi).$$

By the assumption regarding $u_0(y)$, the relationship $\mu(\psi)/w_0(\psi)$ is bounded for $-\infty<\psi<+\infty$. We choose α so large that $-\alpha<\mu(\psi)/w_0(\psi)$. Then for sufficiently large α, sufficiently small X, and $x\leqslant X$ we have

$$\Phi(0,\psi)\leqslant w_N(0,\psi),\qquad \Phi(x,\pm N)\leqslant w_N(x,\pm N), \tag{7a}$$

$$L(\Phi)=\alpha C\exp(-\alpha x)\geqslant|2dp/dx|. \tag{7b}$$

The inequality (6) is easily derived from (7) using the principle of the maximum.

If $dp/dx\leqslant 0$, then for any X and for $\alpha>0$ we obtain $L(\Phi)\geqslant 2dp/dx$ from which (6) can also be derived.

Lemma 2. A solution of problem (3), (5) exists for a certain X in the domain D_N. If $dp/dx\leqslant 0$, then a solution exists for any finite X. All the derivatives of this solution in (3) satisfy Hölder's condition in the closed domain $\bar{D}_N$. In domain D_N Eq.(3) can be differentiated once with respect to x, and twice with respect to ψ.

Proof. If a solution of problem (3), (5) exists then for it an inequality of the form (6) is satisfied. Since the function $\mu(\psi)$ is bounded, $w_N(x,\psi)\geqslant C_0>0$ uniformly in N. let us modify $w_N^{1/2}$ for values of $w_N<C_0$ so that it becomes a smooth positive function. Then Eq.(3) will be a quasilinear parabolic equation in region D_N, and, by Theorem 5.2 in /2, Sect.5, Chapt. VI/, it will have a solution in Hölder's space $H^{2+\beta,1+\beta/2}(\bar{D}_N)$. By Theorem 10 and 11 in /3, Sect.5, Chapt. III/, Eq.(3) can be differentiated inside D_N. The lemma is proved.

Lemma 3. The functions w_N are bounded in D_N uniformly with respect to $N\to\infty$. The derivatives $\partial w_N/\partial\psi$, $\partial w_N/\partial x$, $\partial^2 w_N/\partial\psi^2$ are bounded uniformly in Hölder's norm.

Proof. The uniform boundedness from below of the solutions w_N follows from Lemma 1, and the boundedness from above follows from the principle of the maximum. By Lemma 3.1 from /2, Sect.3, Chapt. VI/, the derivative $\partial w_N/\partial\psi$ is limited uniformly at the boundary of domain D_N. Because of this, $\partial w_N/\partial\psi$ are also bounded in domain D_N uniformly with respect to N, since $z_N=\partial w_N/\partial\psi$ satisfies the parabolic equation

$$\nu w_N^{1/2}\frac{\partial^2 z_N}{\partial\psi^2}+\frac{\nu}{2w_N^{1/2}}z_N\frac{\partial z_N}{\partial\psi}-\frac{\partial z_N}{\partial x}=0.$$

From this point onwards the lemma is proved in the same way as Lemma 7 from /1/.

Theorem 2. In domain D there exists for a certain X a positive bounded solution of problem (3), (4) possessing continuous and bounded derivatives which occur in (3). If $dp/dx\leqslant 0$, then such a solution exists for any finite X.

Proof. It follows from Lemmas 1-3 that we can choose from the family of solutions $\{w_N\}$ a sequence when converges uniformly to any part of domain D together with the derivatives $\partial w_N/\partial x$, $\partial w_N/\partial\psi$, $\partial^2 w_N/\partial\psi^2$. The limit function $w(x,\psi)$ in domain D satisfies equation (3) and the condition $w(0,\psi)=w_0(\psi)$. The boundedness of $w(x,\psi)$ and its derivatives follows from Lemmas 1 and 3.

The proof that $w\to U_1^2(x)$ as $\psi\to+\infty$, and that $w\to U_2^2(x)$ as $\psi\to-\infty$ is analogous to that given regarding the boundary layer in Theorem 2 in /1/.

Theorem 3. the solution of problem (3), (4) which has the properties listed in Theorem 2 is unique.

Proof. We assume that $w_1(x,\psi)$ and $w_2(x,\psi)$ are two solutions of problem (3), (4) which have the properties listed in Theorem 2. For the difference $w_1-w_2\equiv W$ we have

$$\nu w_1^{1/2}\frac{\partial^2 W}{\partial\psi^2}-\frac{\partial W}{\partial x}+\frac{\nu}{w_1^{1/2}+w_2^{1/2}}\frac{\partial^2 w_2}{\partial\psi^2}W=0, \tag{8}$$

and at the same time $W(0,\psi)=0$, $W\to 0$ as $\psi\to+\infty$ and as $\psi\to-\infty$. The coefficient by W in (8) is bounded. Passing from W to $\overline{W}$ in accordance with the formula $W=\overline{W}\exp(\alpha x)$, $\alpha>0$, we obtain

$$\nu w_1^{1/2}\frac{\partial^2\overline{W}}{\partial\psi^2}-\frac{\partial\overline{W}}{\partial x}+\left(\frac{\nu}{w_1^{1/2}+w_2^{1/2}}\frac{\partial^2 w_2}{\partial\psi^2}-\alpha\right)\overline{W}=0,$$

$$\overline{W}(0,\psi)=0,\quad \overline{W}\to 0\quad \text{as}\quad |\psi|\to\infty.$$

We take α so large that

$$\frac{\nu}{w_1^{1/2}+w_2^{1/2}}\frac{\partial^2 w_2}{\partial\psi^2}-\alpha<0.$$

Then, by the principle of the maximum, $\overline{W}$ can have in domain D neither a positive maximum nor a negative minimum. Therefore, $\overline{W}\equiv 0$ in D, that is $W\equiv 0$ and $w_1\equiv w_2$. The theorem is proved.

Theorem 1 is a corollary of Theorems 2 and 3. The proof of this fact is similar to the proof of Lemma 1 in /1/.

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