

Chapter 1

Mathematical preliminaries

In this chapter we quickly review the mathematical preliminaries required for understanding the rest of the material. This is, by no means, an exhaustive chapter in vector calculus. At the end, advanced version of these preliminaries is presented briefly to catalyze further reading. Such advanced discussions are mandatory for developing advanced theories in generalized curvilinear coordinates.

1.1 Index notation

Theoretical development is concise in Cartesian coordinates x, y, z and can be written using a short-hand notation called the index notation. Replacing x, y, z with x_1, x_2, x_3 or x_i where the index i takes values 1, 2, 3. Similarly, in index notation, unit vectors along the axes are denoted by $\hat{e}_i$, components of vector $\mathbf{v}$ by v_i , components of a second order tensor $\mathbf{T}$ by T_{ij} and so on. The need for this short-hand can be motivated by the following example. Consider right-handed orthogonal axis $\hat{e}_1, \hat{e}_2, \hat{e}_3$ and a transformed set $\hat{e}'_1, \hat{e}'_2, \hat{e}'_3$. A vector $\mathbf{v}$ can be denoted as,

$$\mathbf{v} = v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3 = v'_1 \hat{e}'_1 + v'_2 \hat{e}'_2 + v'_3 \hat{e}'_3. \quad (1.1.1)$$

Now, to compute $v'_1 \dots$ components in terms of $v_1 \dots$ components, we will have to write three independent equations taking dot products with $\hat{e}'_1 \dots$ each separately. To avoid this, accepting a few rules of index notation, this relation can be concisely written as simply,

$$v'_i = v_j (\hat{e}_j \cdot \hat{e}'_i). \quad (1.1.2)$$

These rules are listed below with a few examples.

Summation Convention: If an algebraic index (say i) appears exactly twice in a term, then it implies a summation over the range of that index (i) by assigning values 1, 2, 3 to it (i), i.e.,

$$a_i b_i = \sum_{i=1}^3 a_i b_i, \quad \mathbf{v} = v_i \hat{\mathbf{e}}_i, \dots \quad (1.1.3)$$

The repeated algebraic index is called a *dummy index* since it can be replaced by any convenient algebraic index which does not appear in that term. Note that repetition is defined here as the index appearing twice. Hence, the same index appearing thrice in a term or a single expression is ambiguous and not allowed. Hence, Eq. 1.1.2 can be written as all of the following,

$$v'_i = \left(\sum_{j=1}^3 v_j \hat{\mathbf{e}}_j \right) \cdot \hat{\mathbf{e}}'_i = v_j (\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}'_i) = v_k (\hat{\mathbf{e}}_k \cdot \hat{\mathbf{e}}'_i) = v_l (\hat{\mathbf{e}}_l \cdot \hat{\mathbf{e}}'_i) \dots \quad (1.1.4)$$

Same free index on both sides: On both sides of an equation, the free indices must be identical. In Eq. 1.1.2, i is the free index and j is the dummy index. Following are some valid expressions using index notation,

$$a_{ij} b_j = u_{ik} c_k, \quad v_i = a_{ij} b_j, \quad S_{ij} = C_{ijkl} T_{kl}, \dots \quad (1.1.5)$$

Kronecker's delta δ_{ij} is defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \Rightarrow \delta_{11} = \delta_{22} = \delta_{33} = 1, \quad \delta_{12} = \delta_{23} = \delta_{31} = \delta_{21} = \delta_{32} = \delta_{13} = 0 \quad (1.1.6)$$

Note that $\delta_{ij} = \delta_{ji}$ and $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$ is a compact relation for the nine dot-products. If a term with index i is multiplied with δ_{ij} , then index i in the term is substituted by j , since the only non-zero term in the sum is for $i = j$:

$$T_{...i...} \delta_{ij} = \sum_{i=1}^3 T_{...i...} \delta_{ij} = T_{...j...} \quad (1.1.7)$$

For example, $\delta_{ik} a_i = a_k$, $A_{kj} \delta_{ik} = A_{ij}$, $\delta_{ij} \delta_{ij} = \delta_{ii} = 3$. Note that $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$. The **dot-product** of vectors $\mathbf{u} = u_i \hat{\mathbf{e}}_i$ and $\mathbf{v} = v_i \hat{\mathbf{e}}_i$ can be expressed in index form as $\mathbf{u} \cdot \mathbf{v} = u_i v_j \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = u_i v_i$.

1.2 Gauss's theorem

Consider the volume as shown in figure 1.1 with one of its surface projected on $x_1 - x_3$ plane. Consider there is field $\mathbf{u}(\mathbf{x})$ existing in the space and we require to prove,

$$\oint_{\partial S} \mathbf{u} \cdot \hat{\mathbf{n}} dS = \int_V \nabla \cdot \mathbf{u} dV. \quad (1.2.1)$$

Let us only consider u_2 component of the field, for which the above statement of Gauss law takes the form,

$$\int_R u_2 \hat{n}_2 dS = \int_V \frac{\partial u_2}{\partial x_2} dV, \quad (1.2.2)$$

where the surface integration is done on the projected surface R . Consider the RHS of the above equation, which can be written as,

$$\iint \int_{g(x_1, x_3)}^{f(x_1, x_3)} \frac{\partial u_2}{\partial x_2} dx_2 dx_1 dx_3 = \iint (u_2(x_1, f(x_1, x_3), x_3) - u_2(x_1, g(x_1, x_3), x_3)) dx_1 dx_3. \quad (1.2.3)$$

Here, we would like to remark that it is important to consider the total differential as an integrand. Often times, students look at the left hand side of eq. 1.2.1 and think that only quantity immediate to the normal vector needs to be differentiated. In such a case, eq. 1.2.3 will have additional terms due to integration by parts. As we will see below, such terms will show that Gauss's divergence theorem doesn't hold for such a partial derivative.

Consider the area integral now. The normal of a surface given by an implicit equation $F(x_1, x_2, x_3) = 0$ is given by,

$$\hat{\mathbf{n}} = \pm \frac{\nabla F}{|\nabla F|}. \quad (1.2.4)$$

For the two parameterized surfaces, this normal becomes,

$$\hat{\mathbf{n}} = \pm \frac{\frac{\partial f}{\partial x_1} \hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 + \frac{\partial f}{\partial x_3} \hat{\mathbf{e}}_3}{|\nabla(f(x_1, x_3) - x_2)|}. \quad (1.2.5)$$

With the area magnitude, the quantity $\hat{n}_2 dS$ becomes,

$$\hat{n}_2 dS = \pm dx_1 dx_3, \quad (1.2.6)$$

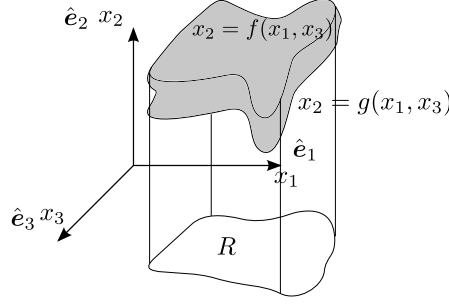


Figure 1.1: A closed volume with parameterized surfaces $x_2 = f$ and $x_2 = g$. Projection of the surface on x_2 plane is shown by R .

where the sign is decided based on top or bottom surface. Substituting in the LHS of equation 1.2.2, we obtain,

$$\iint_{x_2=f(x_1, x_3)} u_2 dx_1 dx_3 - \iint_{x_2=g(x_1, x_3)} u_2 dx_1 dx_3, \quad (1.2.7)$$

which is equal to the RHS of eq. 1.2.3, thus proving the Gauss's divergence theorem for only one component of the field $\mathbf{u}$. Similarly, other components can be shown using the parameterized surfaces of the volume shown.

Using the above, there are following corollaries (not the only ones) of the divergence theorem which maybe useful,

- for $\mathbf{u} \rightarrow g\mathbf{u}$

$$\int_V \nabla \cdot (g\mathbf{u}) dV = \oint_{\partial V} g\mathbf{u} \cdot \hat{\mathbf{n}} dS, \quad (1.2.8)$$

- for $\mathbf{u} \rightarrow \mathbf{c} \times \mathbf{u}$ where $\mathbf{c}$ is a constant field

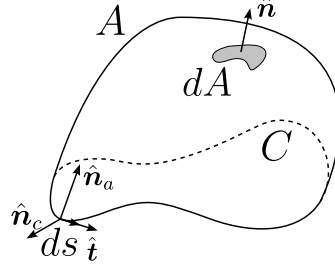
$$\int_V \nabla \cdot (\mathbf{c} \times \mathbf{u}) dV = - \int_V \mathbf{c} \cdot \nabla \times \mathbf{u} dV = \oint_{\partial V} (\mathbf{c} \times \mathbf{u}) \cdot \hat{\mathbf{n}} dS = \oint_{\partial V} \mathbf{c} \cdot \mathbf{u} \times \hat{\mathbf{n}} dS, \quad (1.2.9)$$

or

$$\int_V \nabla \times \mathbf{u} dV = \oint_{\partial V} \mathbf{u} \times \hat{\mathbf{n}} dS, \quad (1.2.10)$$

1.3 Stokes's theorem

In a similar way, Stokes's theorem can be proved which relates the integral along a curve with the integral on the surface bounded by that curve. Consider a curved surface A with bounding curve C in fig. 1.2. Let $\hat{\mathbf{t}}$ be unit tangent vector in direction s along the curve C , $\hat{\mathbf{n}}_a \perp \hat{\mathbf{t}}$ is in the tangent plane of A and $\hat{\mathbf{n}}_c$ is


 Figure 1.2: A bounded surface A by the contour C .

defined as $\hat{\mathbf{t}} \times \hat{\mathbf{n}}_a$. Stokes's theorem states that,

$$\int_C \mathbf{u} \cdot \hat{\mathbf{t}} ds = \int_A (\nabla \times \mathbf{u}) \cdot \hat{\mathbf{n}} dA. \quad (1.3.1)$$

Instead of outlining the full proof (refer to Kreyzig for that), we instead discuss how surface integrals in general are represented. This becomes useful when discussing surface tension. To define a surface integral, take a surface S given by parametric representation as

$$\mathbf{r}(u, v) = x_i(u, v) \hat{\mathbf{e}}_i. \quad (1.3.2)$$

Assume a curve on the surface with parameter t . Then $u(t)$ and $v(t)$ define the curve. Moreover,

$$\frac{d\mathbf{r}(t)}{dt} = \frac{\partial \mathbf{r}}{\partial u} \frac{du}{dt} + \frac{\partial \mathbf{r}}{\partial v} \frac{dv}{dt} = \mathbf{r}_u u' + \mathbf{r}_v v'. \quad (1.3.3)$$

Assuming the partial derivatives $\mathbf{r}_u$ and $\mathbf{r}_v$ are linearly independent (which means u and v are not parallel), the normal vector $\mathbf{N}$ is obtained as $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$. Corresponding unit normal to the surface is defined as,

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}. \quad (1.3.4)$$

The elemental area dS (or dA in figure 1.2) can then be obtained as $dS = |\mathbf{r}_u \times \mathbf{r}_v| du dv$.

1.4 Tensors

Tensors are mathematical objects which are defined using transformation rules under coordinate transforms. For right-handed orthogonal axis $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ and a

transformed set $\hat{\mathbf{e}}'_1, \hat{\mathbf{e}}'_2, \hat{\mathbf{e}}'_3$, let a_{ij} be defined as,

$$a_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j. \quad (1.4.1)$$

Then for an n^{th} order tensor $\mathbf{T}$ with components $\underbrace{T_{ijk\dots}}_n$ defined as,

$$\mathbf{T} = T_{ijk\dots} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k \cdots, \quad (1.4.2)$$

the components transform as,

$$T'_{pqr\dots} = a_{pi} a_{qj} a_{rk} \cdots T_{ijk\dots} \quad (1.4.3)$$

Scalars can be identified as zeroth order tensors which have the same value under coordinate transformation (for example temperature). Vectors can be identified as first order tensors. In fluid mechanics, we often come across second order tensors such as stress, strain rate, velocity gradient etc.

While considering constitutive modeling, we will look at an isotropic fourth order tensor. An isotropic tensor is a tensor whose components remain unaffected under coordinate rotations. By definition, a scalar is an isotropic tensor. An isotropic first order tensor (vector) is $\mathbf{0}$ (a non-zero vector points in some direction so naturally changes with coordinate rotation). A second order isotropic tensor is essentially a diagonal matrix with equal values on the diagonal $A_{ij} = \lambda \delta_{ij}$. A third order isotropic tensor has components proportional to ϵ_{ijk} also known as the permutation symbol. It is defined as,

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123, 231, 312 \text{ cyclic order} \\ -1 & \text{if } ijk = 321, 213, 132 \text{ anti-cyclic order} \\ 0 & \text{if at least two indices are same} \end{cases} \quad (1.4.4)$$

Note the following :-

1. An index on ϵ_{ijk} can be moved two places to right/left without changing its value.
2. For movement of an index by one place to right/left, the sign is reversed.
3. A useful **epsilon-delta identity** is:

$$\epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{lk}. \quad (1.4.5)$$

4. Sometimes, cross-product is used to define the permutation symbol as,

$$\epsilon_{ijk}\hat{\mathbf{e}}_k = \hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j. \quad (1.4.6)$$

5. The components of a vector $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ can be expressed as,

$$\mathbf{c} = c_i \hat{\mathbf{e}}_i = a_j b_k \hat{\mathbf{e}}_j \times \hat{\mathbf{e}}_k = a_j b_k \epsilon_{jki} \hat{\mathbf{e}}_i \rightarrow c_i = \epsilon_{jki} a_j b_k = \epsilon_{ijk} a_j b_k. \quad (1.4.7)$$

The components of a fourth order isotropic tensor A_{ijkl} can be defined using the linear combination of $\delta_{ij}\delta_{kl}$, $\delta_{il}\delta_{jk}$, and $\delta_{ik}\delta_{jl}$.

1.4.1 Tensor calculus

The most useful tool of tensor calculus for us is the Nabla operator defined using index notation as,

$$\nabla = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}. \quad (1.4.8)$$

Note that the above definition is naturally restricted to the Cartesian coordinate systems. In cylindrical coordinates, the definition is,

$$\nabla = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\phi}{r} \frac{\partial}{\partial \phi} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}. \quad (1.4.9)$$

Note that $\frac{\partial \hat{\mathbf{e}}_r}{\partial \phi} = \hat{\mathbf{e}}_\phi$ and $\frac{\partial \hat{\mathbf{e}}_\phi}{\partial r} = -\hat{\mathbf{e}}_r$.

The gradient of a field $f(\mathbf{x})$ is defined as ∇f .

- For a scalar field f , $\nabla f = \frac{\partial f}{\partial x_i} \hat{\mathbf{e}}_i$. The reader is encouraged to show that it transforms as a vector.
- For a vector field $\mathbf{f}$, $\nabla \mathbf{f} = \frac{\partial f_j}{\partial x_i} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j = \frac{\partial f_j}{\partial x_i} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j$. The reader is encouraged to show that it transforms as a second order tensor

For tensor fields of order equal or higher than 1, divergence and curl are defined as,

1. Divergence : $\nabla \cdot \mathbf{T} = \frac{\partial T_{ijk\dots}}{\partial x_p} \hat{\mathbf{e}}_p \cdot \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k \dots = \frac{\partial T_{ijk\dots}}{\partial x_i} \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k \dots$
2. Curl : $\nabla \times \mathbf{T} = \frac{\partial T_{ijk\dots}}{\partial x_p} \hat{\mathbf{e}}_p \times \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k \dots = \epsilon_{piq} \frac{\partial T_{ijk\dots}}{\partial x_p} \hat{\mathbf{e}}_q \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k \dots$

Fields for which the divergence is 0, $\nabla \cdot \mathbf{T} = 0$ are called *solenoidal*. Similarly, when $\nabla \times \mathbf{T} = 0$, the field $\mathbf{T}$ is called *irrotational*. There are further classifications

of fields (specifically for vectors),

1. If $\nabla \cdot \mathbf{a} = 0$ and $\nabla \times \mathbf{a} = 0$, then $\mathbf{a} = \nabla \phi$ where ϕ is a potential function.
2. Complex lamellar field : $\mathbf{a} \cdot (\nabla \times \mathbf{a}) = 0$. (Note that all vector fields are not complex lamellar or the curl is not always normal to the vector field).
3. Beltrami field : $\mathbf{a} \times (\nabla \times \mathbf{a}) = 0$

According to *Helmholtz' theorem*, a continuous vector field which vanishes at infinity can always be written using three scalar functions as ,

$$\mathbf{a} = \nabla \phi + \nabla \times (\psi \nabla \chi), \quad (1.4.10)$$

or a *scalar potential function* and a *vector potential function*,

$$\mathbf{a} = \nabla \phi + \nabla \times \mathbf{A}. \quad (1.4.11)$$

For a tensor field (of any order), Taylor's expansion about a point $\mathbf{x}_0$ in the direction $\hat{n}$ yields,

$$\mathbf{T}(\mathbf{x}_0 + \hat{n}dr) = \mathbf{T}(\mathbf{x}_0) + \hat{n}dr \cdot \nabla \mathbf{T} \Big|_{\mathbf{x}_0} + \frac{1}{2} \nabla (\nabla \mathbf{T}) \Big|_{\mathbf{x}_0} : \hat{n} \otimes \hat{n} (dr^2) + \dots, \quad (1.4.12)$$

In the limit $dr \rightarrow 0$, the above expression shows that $\hat{n} \cdot \nabla \mathbf{T} \Big|_{\mathbf{x}_0}$ is the directional derivative of the field $\mathbf{T}$ at $\mathbf{x}_0$ in direction $\hat{n}$.

Often, derivatives of the form $\mathbf{u} \cdot \nabla$ appear in fluid mechanics where $\mathbf{u}$ is the fluid velocity. This can be interpreted as a directional derivative along the fluid velocity or the streamlines (curves always parallel to velocity directions).

Advanced tensor calculus 1.1

Optional

Tensor calculus is a vast field. One of the key assumptions made in undergraduate courses is that the unit vectors $\hat{e}_i$ are constant in space. In generalized curvilinear coordinates, this is not the case. Two ubiquitous examples of curvilinear coordinates are the cylindrical coordinates and the spherical coordinates. Specifically, these are examples of orthogonal curvilinear coordinates. In this box, we will work out a sample calculation using generalized orthogonal curvilinear coordinates. For non-orthogonal coordinates, the reader is referred to [Aris \[2012\]](#).

1.5 Advanced summation convention

Any index repeated once in the upper position and once in the lower position in a product is called a dummy index and summation over its values is implied. Any index not repeated is called the free index.

Additionally, any free index must appear in the same position in each term of an equation. Formula

$$A_k^{ij} = B_{ip} C_q^{pj} D^q, \quad (1.5.1)$$

is invalid.

The most important rule of this advanced index notation is that an upper affix in a denominator counts as a lower index in the numerator and vice-versa. This $D_i = \frac{\partial}{\partial x^i}$ is a valid symbol.

1.6 Contravariant and covariant vectors

Let x^j be coordinates and ξ^i be transformed (general curvilinear coordinates). The differentials are related as,

$$d\xi^i = \frac{\partial \xi^i}{\partial x^j} dx^j. \quad (1.6.1)$$

These differentials are termed as the *contravariant* vectors. Any vector components a^i are called the components of a contravariant vector at a certain point in the x^j coordinate system, if under a transformation to ξ^i coordinate system, the components become,

$$\tilde{a}^i = \frac{\partial \xi^i}{\partial x^j} a^j. \quad (1.6.2)$$

For a moving particle, the velocity components $v^i = dx^i/dt$ transform as $\tilde{v}^i = d\tilde{x}^i/dt$ which are related as,

$$\tilde{v}^i = \frac{d\tilde{x}^i}{dt} = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{dx^j}{dt} = \frac{\partial \tilde{x}^i}{\partial x^j} v^j. \quad (1.6.3)$$

Hence, velocity is a contravariant vector.

For a scalar function $f(x^1, x^2, x^3)$, derivative transform according to the equation,

$$\frac{\partial f}{\partial \tilde{x}^i} = \frac{\partial x^j}{\partial \tilde{x}^i} \frac{\partial f}{\partial x^j}, \quad (1.6.4)$$

or

$$\tilde{a}_i = \frac{\partial x^j}{\partial \tilde{x}^i} a_j. \quad (1.6.5)$$

In this case, a_i are components of a covariant vector in the coordinate system x^i , since they transform according to Eq. (1.6.5). An astute reader would have already noticed that the affix in covariant components is a lower affix, since the transformation of these vectors is similar to vectors defined through scalar gradients. In an intuitive sense, contravariant vectors transform as tangents to curve would transform and covariant vectors transform as normals (gradients) would transform. An example of a covariant vector is the basis vectors. Let $\mathbf{y} = y^1, y^2, y^3$ be the Cartesian coordinates of a point and x^1, x^2, x^3 its coordinates in some other curvilinear system. The quantities $\frac{\partial y^i}{\partial x^j}$ for fixed i are the components of a covariant vector. For fixed j ,

$$\mathbf{g}_{(j)} = \frac{\partial \mathbf{y}}{\partial x^j}, \text{ (notice the lower index)} \quad (1.6.6)$$

is a Cartesian vector. The three Cartesian vectors $\mathbf{g}_{(1)}, \mathbf{g}_{(2)}, \mathbf{g}_{(3)}$ are called base vectors in the coordinate system x^j . Any Cartesian vector can be expressed as,

$$\mathbf{a} = a^1 \mathbf{g}_{(1)} + a^2 \mathbf{g}_{(2)} + a^3 \mathbf{g}_{(3)} = a^i \mathbf{g}_{(i)}. \quad (1.6.7)$$

For another system of coordinates $\tilde{x}^k$, we would have base vectors $\tilde{\mathbf{g}}_{(k)}$ defined. The reader is encouraged to show through transformation of a^i that they are components of a contravariant vector.

1.7 The metric tensor

Let the distance between two points A and B with coordinates y^i and $y^i + dy^i$ respectively, be ds . Then,

$$ds^2 = \sum_i dy^i dy^i. \quad (1.7.1)$$

Note that no summation is implied when both indices are upper (or both are lower). Hence an explicit summation sign is used. Using another coordinate system x^j ,

$$ds^2 = \sum_i \left(\frac{\partial y^i}{\partial x^j} dx^j \right) \left(\frac{\partial y^i}{\partial x^k} dx^k \right) = g_{jk} dx^j dx^k, \quad (1.7.2)$$

where

$$g_{ij} = \sum_k \frac{\partial y^k}{\partial x^i} \frac{\partial y^k}{\partial x^j} \quad (1.7.3)$$

is the metric tensor. A *generalized curvilinear coordinate system* has only diagonal terms nonzero in the metric tensor, and it is often convenient to write the three independent diagonal components as,

$$g_{11} = h_1^2, \quad g_{22} = h_2^2, \quad g_{33} = h_3^2, \quad (1.7.4)$$

where h_i are called the scale factors. For an infinitesimal displacement in the direction of the x^i coordinate, $ds = h_i dx^i$ (no summation). Cartesian coordinates all have the physical dimensions of length. In general, this is not the case for curvilinear coordinates. Hence, contravariant velocity components $v^i = dx^i/dt$ would not all have the same physical dimensions. For a general orthogonal curvilinear system,

$$|A|^2 = (h_1 A^1)^2 + (h_2 A^2)^2 + (h_3 A^3)^2, \quad (1.7.5)$$

so that $h_i A^i$ have the same physical dimensions as the magnitude of A . Similar to the basis vectors $\mathbf{g}_{(i)}$, we can define the reciprocal basis vectors $\mathbf{g}^{(i)}$,

$$\mathbf{g}^{(i)} = \frac{\mathbf{g}_{(j)} \times \mathbf{g}_{(k)}}{\sqrt{g}}, \quad (1.7.6)$$

where g is the determinant of the metric tensor and $\mathbf{g}^{(i)} \cdot \mathbf{g}_{(j)} = \delta_j^i$ is satisfied. Analogously, we can define the *conjugate metric tensor* as,

$$g^{ij} = \mathbf{g}^{(i)} \cdot \mathbf{g}^{(j)} \quad (1.7.7)$$

The corresponding covariant components of $\mathbf{a}$ can be defined using $\mathbf{a} = a_k \mathbf{g}^{(k)}$. The relation between the contravariant components and covariant components of a vector can be obtained using,

$$\left(a^i \mathbf{g}_{(i)}\right) \cdot \mathbf{g}^{(k)} = a_j \mathbf{g}^{(j)} \cdot \mathbf{g}^{(k)}, \quad (1.7.8)$$

$$\implies a^i \delta_i^k = a_j g^{jk} \implies a^k = g^{jk} a_j. \quad (1.7.9)$$

1.8 Derivatives and Christoffel symbols

Consider a contravariant vector $\mathbf{a} = a^i \mathbf{g}_{(i)}$. The derivative w.r.t. $x^{(k)}$ (Covariant differentiation) is given by,

$$\frac{\partial \mathbf{a}}{\partial x^k} = \frac{\partial a^i}{\partial x^k} \mathbf{g}_{(i)} + a^i \frac{\partial \mathbf{g}_{(i)}}{\partial x^k}. \quad (1.8.1)$$

The derivative of the basis vectors w.r.t. the curvilinear coordinates, expressed in terms of the basis vectors are known as the Christoffel symbols (for basis vectors, these are Christoffel symbols of second kind) and can be written as,

$$\Gamma_{jk}^i \mathbf{g}_{(i)} = \frac{\partial \mathbf{g}_{(j)}}{\partial x^k}. \quad (1.8.2)$$

Now, we are in a position to define gradient, Laplacian, divergence, and curl of tensorial quantities in generalized curvilinear coordinates.

Gradient of a scalar field is by definition a covariant vector (since the transformation happens like a normal). For a scalar function ϕ , the gradients $\partial\phi/\partial x^i$ can be written in short-hand like $\phi_{,i}$ which are covariant components. The vector is given by,

$$\nabla\phi = \phi_{,i} \mathbf{g}^{(i)}, \quad (1.8.3)$$

in which summation is implied as per rules. The reader is urged to work this out for cylindrical coordinates and confirm. The corresponding contravariant components of the gradient are $g^{ij}\phi_{,i}$. To define Laplacian, we

Optional

differentiate this with x^j to get,

$$\frac{\partial}{\partial x^j} \left(g^{ij} \frac{\partial \phi}{\partial x^i} \right) + \Gamma_{jk}^j g^{ik} \frac{\partial \phi}{\partial x^i}, \quad (1.8.4)$$

which can be further simplified to familiar forms. The reader should refer to Chapter 7 in [Aris \[2012\]](#) for full description.

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