

Chapter 2

Continuum Kinematics of Fluids

In this chapter, we review the basics of continuum mechanics required to describe the kinematics of fluid flows. Since kinematics refers to the study of motion without describing the forces/stresses causing the motion, we will resort to some basic terminology used in continuum mechanics, given that we are modeling fluids as continua.

While modeling continua, we represent a material body by a continuous distribution of an infinite number of continuum particles. Each continuum particle itself maybe considered as a combination of countably infinite (think of total number of integers) molecules of the material. Yet, each continuum particle is treated as a geometrical point in space. Visually, as shown in figure 2.1, the length scale ℓ of the continuum particle which may be a few thousands to a few millions of mean-free-path lengths itself, is ignored compared to the overall length scale of the material under study. In doing so, we make the assumption that *material or specimen under study is made up of uncountably infinite (think of total number of real numbers) continuum particles*.

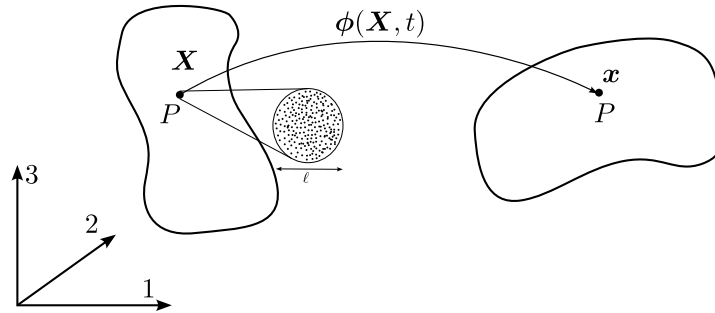


Figure 2.1: A continuum particle P in reference and deformed configurations.

Continuum approximation 2.1**Detail**

A more thorough understanding of the concept of continuum assumption can be obtained by considering statistical mechanics [Bird, 1994]. For a system of N molecules in a volume of V , the probability of finding n molecules in a sub-volume v is given by,

$$P_N(n) = \binom{N}{n} \left(\frac{v}{V}\right)^n \left(1 - \frac{v}{V}\right)^{N-n}. \quad (2.0.1)$$

For very large number of total molecules $N \gg 1$ and a small sub-volume $v/V \ll 1$ such that

$$v \frac{N}{V} \left(1 - \frac{v}{V}\right) \approx v \frac{N}{V} \gg 1, \quad (2.0.2)$$

the binomial distribution in Eq. (2.0.1) can be approximated using the so-called DeMoivre-Laplace equation as,

$$P_N(n) \approx \frac{e^{-\frac{\left(n - \frac{N}{V}v\right)^2}{2v \frac{N}{V}}}}{\sqrt{2\pi \frac{N}{V}v}}, \quad (2.0.3)$$

which is a normal distribution. For number density ρ_n , the above expression can be further simplified. This shows that the standard deviation from the mean value $\rho_n v$ is $\sqrt{\rho_n v}$. Relative to the mean value, it is $1/\sqrt{\rho_n v}$ which is very small. Hence, in a way, Eq. (2.0.2) can be seen as a mathematical representation of the continuum approximation, as described above this detail box.

2.1 Deformation mapping

As a continuum body changes configuration from the reference configuration to some deformed configuration in time t , all the continuum particles or points (P) in the reference configuration get mapped to their positions in the deformed configuration, as shown in figure 2.1. If we identify the continuum particle P by $\mathbf{X}$ and its position in the deformed configuration by $\mathbf{x}$, then we can define a mapping ϕ as,

$$\mathbf{x}(\mathbf{X}, t) = \phi(\mathbf{X}, t). \quad (2.1.1)$$

In standard continuum mechanics literature ϕ is called the deformation map defined over all the continuum points in the reference configuration. Note that $\mathbf{x}$ is a position vector, hence ϕ is a vector valued function.

2.1.1 Lagrangian and Eulerian description

In continuum mechanics, we identify fields which are properties associated to the continuum particles or points. These fields may be written as functions of the reference configuration coordinates $\mathbf{X}$ or the deformed configuration coordinates $\mathbf{x}$. For instance, consider a scalar field T such as temperature. We can write T as,

$$T = T(\mathbf{x}, t) \text{ or } T = \mathbb{T}(\mathbf{X}, t), \quad (2.1.2)$$

where

$$\mathbb{T}(\mathbf{X}, t) = T(\phi(\mathbf{X}, t), t). \quad (2.1.3)$$

Writing as $T(\mathbf{x}, t)$, the temperature field is written in terms of spatial position $\mathbf{x}$. Hence, such a description is known as the spatial description or *Eulerian approach*. On the other hand, $\mathbb{T}(\mathbf{X}, t)$ writes the field in terms of material points (in reference configuration) and hence is called material description or *Lagrangian approach*. In fluid mechanics, Eulerian approach is usual while in solid mechanics, Lagrangian approach is often preferred. Primarily, because material particles can travel large relative distances in fluid flows, hence the concept of reference configuration has very little meaning.

2.1.2 Spatial tensor fields

While it is possible to define material tensor fields using $\mathbf{X}$ (Lagrangian coordinates), for this course, we focus on the spatial tensor fields, defined on the spatial coordinate $\mathbf{x}$. General spatial tensor fields $\mathbf{T}(\mathbf{x}, t)$ can be associated with a location $\mathbf{x}$. Note that these are still physical properties. The visually way of reading $\mathbf{T}(\mathbf{x}, t)$ is: *property $\mathbf{T}$ of particles which are at $\mathbf{x}$ at time t* . Similarly, derivatives w.r.t. spatial coordinate $\mathbf{x}$ can be written for these fields using the nabla operator defined in spatial coordinates, as reviewed in previous chapter. For derivatives of material tensor fields, we use the notation ∇_0 defined as,

$$\nabla_0 = \hat{\mathbf{E}}_i \frac{\partial}{\partial X_i}, \quad (2.1.4)$$

where we have used $\hat{\mathbf{E}}_i$ to denote unit vectors used to describe material coordinates. We have defined a material tensor field already, that is, the deformation map $\phi(\mathbf{X}, t)$.

2.1.3 Deformation gradient

The gradient of the deformation map is known as the *deformation gradient*, usually denoted by $\mathbf{F}(\mathbf{X}, t)$. Since the deformation map is a vector valued function, deformation gradient is a second order tensor defined as,

$$\mathbf{F}(\mathbf{X}, t) = \nabla_0 \phi = \nabla_0 \mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}. \quad (2.1.5)$$

Usually, a tensor containing both spatial and material coordinates is known as a two point tensor. In a visual sense, components of the deformation gradient F_{iJ} can be written as a matrix. Capital index $(\)_J$ has been used to highlight that the second index is assumed to run over material coordinates. The deformation gradient $\mathbf{F}$ encapsulates the information of the change in line elements, area elements, and volume elements as the material undergoes deformation (see [Tadmor and Miller \[2011\]](#)). For fluid kinematics, we only need to look at the volume deformation, which is encapsulated in the determinant of the matrix represented by the deformation gradient. Consider $d\mathbf{x}$ to be an infinitesimal vector in the deformed configuration, obtained by deforming $d\mathbf{X}$. Then, by definition,

$$dx_i = F_{iJ} dX_J. \quad (2.1.6)$$

Note that the block index J above denotes that the second index corresponds to the material coordinates. Consider three material element vectors $d\mathbf{X}, d\mathbf{Y}, d\mathbf{Z}$ making a parallelepiped. The volume of this parallelepiped is $dV_0 = |d\mathbf{Z} \cdot (d\mathbf{X} \times d\mathbf{Y})|$. Similarly, the volume of the deformed version of this parallelepiped is $dV = |d\mathbf{z} \cdot (d\mathbf{x} \times d\mathbf{y})|$. Using index notation, we write

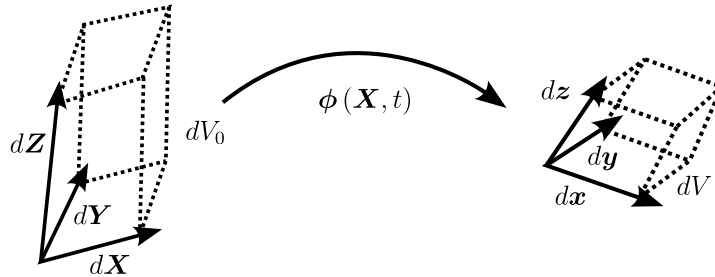


Figure 2.2: Deformation of a volume element.

$$dV = |\epsilon_{ijk} dx_i dy_j dz_k| = |\epsilon_{ijk} F_{iI} F_{jJ} F_{kK} dX_I dY_J dZ_K| \quad (2.1.7)$$

In index notation, determinant of a general matrix A can be written as,

$$\det(A) = \epsilon_{ijk} A_{i1} A_{j2} A_{k3}. \quad (2.1.8)$$

Additionally, the quantity $\epsilon_{ijk} A_{ip} A_{jq} A_{kr}$ is a completely skew-symmetric matrix in p, q, r hence can be written as

$$\epsilon_{ijk} A_{ip} A_{jq} A_{kr} = C \epsilon_{pqr}, \quad (2.1.9)$$

where $C = \det(A)$. Hence, the deformed volume dV simplifies to,

$$dV = |\det(F)| |\epsilon_{IJK} dX_I dY_J dZ_K| = |\det F| dV_0. \quad (2.1.10)$$

For a continuous deformation which is invertible, matrix representing $\mathbf{F}$ is not allowed to be singular. Hence the determinant is not allowed to go to 0. Since $\det(F) = 1$ initially, $\det(F) > 0$ throughout deformation. We will denote it by $\mathcal{J}$. Additional relations between $\mathcal{J}$ and $[F^{-1}]$ are,

$$\mathcal{J} = \frac{1}{6} \epsilon_{IJK} \epsilon_{ijk} F_{iI} F_{jJ} F_{kK}, \quad [F^{-1}]_{Ji} = \frac{1}{2\mathcal{J}} \epsilon_{ipq} \epsilon_{JKL} F_{pK} F_{qL}. \quad (2.1.11)$$

Using the above, the derivative of the determinant w.r.t. the components of the matrix can be simplified to,

$$\frac{\partial \mathcal{J}}{\partial F_{iJ}} = \mathcal{J} [F^{-1}]_{Ji}, \quad (2.1.12)$$

which may be written as

$$\frac{\partial \mathcal{J}}{\partial \mathbf{F}} = \mathcal{J} \mathbf{F}^{-T}. \quad (2.1.13)$$

Additionally, for a relative position vector $\mathbf{M}$ in the reference configuration, the relative position vector $\mathbf{m}$ in the deformed configuration is given by $\mathbf{m} = \mathbf{F}\mathbf{M}$.

2.2 Rate of change and velocity gradient

While calculating rate of change of quantities, it is essential to distinguish if the quantities are written as functions of spatial coordinates or material coordinates. Returning to our variables T and $\mathbb{T}$, defined in spatial and material coordinated respectively, we define the material time derivative as

$$\dot{T} = \left. \frac{\partial \mathbb{T}}{\partial t} \right|_{\mathbf{x}} = \frac{DT}{Dt}. \quad (2.2.1)$$

Essentially, material time derivative denotes the rate of change of quantity $\mathbb{T}$ associated to a fixed material particle, identified by $\mathbf{X}$. Let the quantity be the position vector of the material particle itself. Then, $\mathbb{T} = \phi(\mathbf{X}, t)$. In that case, the velocity $\mathbb{V}(\mathbf{X}, t)$ is calculated as,

$$\mathbb{V}(\mathbf{X}, t) = \dot{\mathbf{x}} = \frac{\partial \phi}{\partial t}. \quad (2.2.2)$$

Similarly, the acceleration of the material particle is given by $\mathbb{A} = \ddot{\mathbf{x}}$. The velocity and acceleration in the spatial description are,

$$\mathbb{V}(\mathbf{X}, t) = \mathbf{u}(\mathbf{x}, t), \quad \mathbb{A} = \mathbf{a}(\mathbf{x}, t). \quad (2.2.3)$$

In fluid mechanics, we need to denote/calculate, material time derivatives of spatial fields. Noting that $\mathbf{x}$ is changing in time, we obtain,

$$\frac{DT}{Dt} = \frac{DT(\mathbf{x}, t)}{Dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x_i} \frac{\partial x_i}{\partial t} = \frac{\partial T}{\partial t} + \mathbf{u}(\mathbf{x}, t) \cdot \nabla T. \quad (2.2.4)$$

Note that the above defines the rate of change of T for a fixed material particle identified by $\mathbf{X}$, as its $\mathbf{x}$ changes in time. Similarly, components of acceleration $\mathbf{a}$ can be written as,

$$\frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j} \frac{\partial x_j}{\partial t}. \quad (2.2.5)$$

In the vector notation, this can be written as,

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}, \quad (2.2.6)$$

where, the gradient of velocity is defined as,

$$\nabla \mathbf{u} = \frac{\partial u_i}{\partial x_j} \hat{\mathbf{e}}_j \hat{\mathbf{e}}_i. \quad (2.2.7)$$

Before we derive the physical interpretation of the velocity gradient tensor, we can show that the local deformation gradient rate $\dot{\mathbf{F}}$ is also related to the velocity gradient tensor. Since $d\mathbf{x}$ is a function of $\mathbf{X}$ and time t , time rate of change of $d\mathbf{x}$ can be obtained as,

$$dx_i = \dot{F}_{iJ} dX_J. \quad (2.2.8)$$

The time rate of change of the components of the deformation gradient $\mathbf{F}$ can be evaluated as,

$$\dot{F}_{iJ} = \frac{\partial \dot{x}_i}{\partial X_J} = \frac{\partial u_i}{\partial x_j} \frac{\partial x_j}{\partial X_J}. \quad (2.2.9)$$

Since $dx_i(\mathbf{X}, t)$, rate of change of dx_i yields the velocity of the material particles in the material description (denoting as $\mathbb{V}$) which can be written as $\mathbf{u}$ as a function of spatial coordinates. The above shows that,

$$\begin{aligned}\dot{\mathbf{F}} &= \mathbf{F} \cdot \nabla \mathbf{u} = \frac{\partial x_i}{\partial X_J} \hat{\mathbf{E}}_J \hat{\mathbf{e}}_i \cdot \frac{\partial u_p}{\partial x_q} \hat{\mathbf{e}}_q \hat{\mathbf{e}}_p = \frac{\partial x_i}{\partial X_J} \frac{\partial u_p}{\partial x_q} \hat{\mathbf{E}}_J \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_q \hat{\mathbf{e}}_p = \frac{\partial x_i}{\partial X_J} \frac{\partial u_p}{\partial x_q} \hat{\mathbf{E}}_J \delta_{iq} \hat{\mathbf{e}}_p \\ &= \frac{\partial x_i}{\partial X_J} \frac{\partial u_p}{\partial x_i} \hat{\mathbf{E}}_J \hat{\mathbf{e}}_p.\end{aligned}\tag{2.2.10}$$

2.2.1 Velocity gradient decomposition

The velocity gradient tensor can be written as a sum of a symmetric second order tensor and an antisymmetric second order tensor as,

$$\nabla \mathbf{u} = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + \frac{1}{2} (\nabla \mathbf{u} - (\nabla \mathbf{u})^T) = \mathbf{S} + \mathbf{\Omega}.\tag{2.2.11}$$

$\mathbf{S}$ is called the strain rate tensor (or simply strain rate) and $\mathbf{\Omega}$ is called the rotation rate tensor (or simply rotation rate). This physical interpretation can be obtained as follows. Consider two material particles in the deformed configuration separated by $d\mathbf{x}$. The squared length between the two particles is,

$$ds^2 = d\mathbf{x}_i d\mathbf{x}_i.\tag{2.2.12}$$

The rate of change of this length can be evaluated as,

$$2ds\dot{ds} = 2d\mathbf{x}_i \dot{d\mathbf{x}}_i = 2 \frac{\partial v_i}{\partial x_j} d\mathbf{x}_j d\mathbf{x}_i.\tag{2.2.13}$$

Since the tensor with components $d\mathbf{x}_i d\mathbf{x}_j$ is symmetric, the above expression simplifies to,

$$\dot{ds} = \frac{1}{ds} S_{ij} d\mathbf{x}_i d\mathbf{x}_j = \frac{1}{ds} \mathbf{S} : (d\mathbf{x} \otimes d\mathbf{x}).\tag{2.2.14}$$

Hence, $\mathbf{S}$ accounts for the rate of deformation (or rate of change of lengths between two material particles). In particular, $\mathbf{S}$ is the rate of strain. To bring out this intuition, we divide the above equation by ds^2 to obtain,

$$\frac{1}{ds} \frac{d}{dt}(ds) = S_{ij} \frac{dx_i}{ds} \frac{dx_j}{ds}.\tag{2.2.15}$$

Since dx_i/ds simply represents the components of unit vector in the direction of $d\mathbf{x}$, the left hand side represents the rate of strain. Since $\mathbf{S}$ is a second or-

der symmetric tensor, it can be represented as a symmetric matrix. It is easily shown that a real symmetric matrix has three mutually perpendicular eigenvectors. Using these, the strain rate tensor can be diagonalized as well. These three mutually independent directions are called the principal axes of strain rate and the corresponding diagonal elements in the diagonalized tensor are the principal rates of strain.

2.3 Vorticity

To see the rotation rate interpretation of $\mathbf{\Omega}$, we recognise that any antisymmetric tensor can be written as,

$$\Omega_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_k \quad (2.3.1)$$

for some vector $\boldsymbol{\omega}$. Note that an antisymmetric second order tensor can have only three independent components (antisymmetric matrix). Hence, those components can be arranged as components of a hypothetical vector. This hypothetical vector is called the dual vector of the antisymmetric tensor. The physical meaning of the dual vector of $\mathbf{\Omega}$ becomes clear as we consider the relative velocity between two material points at spatial locations separated by $d\mathbf{x}$ given by,

$$d\mathbf{u} = d\mathbf{x} \cdot \nabla \mathbf{u} = dx_j \hat{\mathbf{e}}_j \cdot \frac{\partial u_i}{\partial x_k} \hat{\mathbf{e}}_k \hat{\mathbf{e}}_i \implies du_i = \frac{\partial u_i}{\partial x_j} dx_j. \quad (2.3.2)$$

Substituting the decomposition we get,

$$du_i = S_{ij}dx_j + \Omega_{ij}dx_j = S_{ij}dx_j - \frac{1}{2}\epsilon_{ijk}\omega_k dx_j = S_{ij}dx_j + \epsilon_{ijk}\frac{1}{2}\omega_j dx_k, \quad (2.3.3)$$

which in the vector rotation gives,

$$d\mathbf{u} = \mathbf{S} \cdot d\mathbf{x} + \left(\frac{1}{2}\boldsymbol{\omega}\right) \times d\mathbf{x}. \quad (2.3.4)$$

The above expression makes clear that the dual vector of the rotation rate tensor $\boldsymbol{\omega}$ is equal to twice the relative rotation of two material particles separated by a vector $d\mathbf{x}$ in the deformed configuration. For definition of $\boldsymbol{\omega}$, we invert the dual vector relation in Eq. 2.3.1 as

$$\Omega_{ij}\epsilon_{lik} = -\frac{1}{2}\epsilon_{ijk}\epsilon_{lij}\omega_k, \quad (2.3.5)$$

which yields,

$$\omega_l = -\epsilon_{lij}\Omega_{ij} = -\epsilon_{lij}\frac{\partial u_i}{\partial x_j} = \epsilon_{lij}\frac{\partial u_j}{\partial x_i} \implies \boldsymbol{\omega} = \nabla \times \mathbf{u}. \quad (2.3.6)$$

$\boldsymbol{\omega}$ is called the vorticity vector and is defined as the curl of velocity. By definition, an *irrotational flow* has zero vorticity everywhere. For irrotational flows, for the particular orientation of the coordinate axis along the eigenvectors of $\mathbf{S}$, the transformed $\mathbf{S}$ is a diagonal matrix yielding,

$$du_1 = S_{11}dx_1, \quad du_2 = S_{22}dx_2, \quad du_3 = S_{33}dx_3. \quad (2.3.7)$$

This highlights that a hypothetical sphere in an irrotational flow will get stretched into an ellipsoid, the axis of which are oriented along the eigenvectors (principal axes of deformation) of $\mathbf{S}$.

The trajectories (lines tangential to vortex vectors) of the vortex field are called the vortex lines and the surface generated by the vortex lines through a closed curve C is a *vortex tube*. The *strength* of a tube of the vector field $\mathbf{a}$ at its section S is defined as,

$$\iint_S \mathbf{a} \cdot \mathbf{n} dS. \quad (2.3.8)$$

We see that the strength of a vortex tube equals the *circulation* around the closed curve C which bounds the cross-section S ,

$$\iint_S (\nabla \times \mathbf{u}) \cdot \mathbf{n} dS = \oint_C \mathbf{u} \cdot \mathbf{t} ds = \Gamma, \quad (2.3.9)$$

where $\mathbf{t}$ is the unit vector tangent to the curve C and Γ is called the total circulation.

2.3.1 Induced velocity

A very interesting part of fluid kinematics is obtaining the reverse, i.e., *calculating the velocity field which yields a prescribed vorticity field*. To this end, we consider the Helmholtz' decomposition of the velocity field $\mathbf{u}$ and write,

$$\mathbf{u} = \nabla\phi + \nabla \times \boldsymbol{\Psi}. \quad (2.3.10)$$

Taking divergence and curl of the above readily shows that ϕ is responsible for velocity divergence and $\boldsymbol{\Psi}$ is responsible for the vorticity. Hence the above decomposition may be intuitively considered as a summation of a *dilatational*

velocity field $\mathbf{u}_d$ and an *induced vortical* velocity field $\mathbf{u}_v$. Below, we will see how to compute $\mathbf{u}_v$ from a known $\nabla \times \mathbf{u} = \boldsymbol{\omega}$.

Taking a curl of Eq. (2.3.10) we obtain,

$$\boldsymbol{\omega} = \nabla \times (\nabla \times \boldsymbol{\Psi}) = \nabla (\nabla \cdot \boldsymbol{\Psi}) - \nabla^2 \boldsymbol{\Psi}. \quad (2.3.11)$$

Evaluation of a unique $\boldsymbol{\Psi}$ is not possible from Eq. (2.3.10), since its a differential equation in $\boldsymbol{\Psi}$ without any boundary condition. So we fix that $\nabla \cdot \boldsymbol{\Psi} = 0$. This may be interpreted as evaluating $\boldsymbol{\Psi}$ with an arbitrary divergence. Hence, the equation to solve is,

$$\boldsymbol{\omega} = -\nabla^2 \boldsymbol{\Psi}. \quad (2.3.12)$$

We suppose that we are trying to find $\boldsymbol{\Psi}$ in an infinite space, far away from any physical boundaries. The solution is then provided by the Green's function in $\mathbb{R}^3$ as,

$$\boldsymbol{\Psi}(\mathbf{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\boldsymbol{\omega}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y}, \quad (2.3.13)$$

which then yields the induced velocity expression as,

$$\mathbf{u}_v = \nabla \times \boldsymbol{\Psi} = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\boldsymbol{\omega}(\mathbf{y}) \times (\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^3} d\mathbf{y}. \quad (2.3.14)$$

The expression of induced velocity (or, the velocity required to maintain a given vorticity) is extremely intuitive. It shows the contribution of velocity at point $\mathbf{x}$ due to vorticity at all other points $\mathbf{y}$. The expression Eq. (2.3.14) can be used

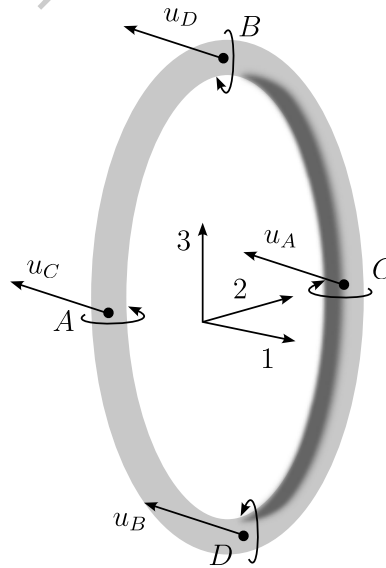


Figure 2.3: Induced velocity due to a vortex ring.

to find the propagation speed of a vortex ring. A vortex ring is a collection of vortices arranged over a ring, such that their rotational direction maintains the clockwise or counter-clockwise sense as shown. Consider 4 diametrically opposite ends, as shown schematically in the figure. Each point *induces* a velocity in negative 1 direction on its diametrically opposite end. Consequently, by virtue of its rotation, the vortex ring can not remain stationary in an empty medium and it propagates in a particular direction, depending on the rotational sense of the infinitesimal vortices arranged.

Green's function of the Poisson equation 2.2

Detail

To derive the Green's function, we may consider the fundamental solution of the Laplace equation, or proceed using the Fourier transforms. The Green's function solves the equation,

$$\nabla^2 G = \delta(\mathbf{x} - \mathbf{y}), \quad (2.3.15)$$

where

$$\delta(\mathbf{x} - \mathbf{y}) = \frac{1}{8\pi^3} \int_{\mathbb{R}^3} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} d\mathbf{k}, \quad (2.3.16)$$

is the Dirac delta function with the properties

$$\int_{\mathbb{R}^3} \delta(\mathbf{x}) d\mathbf{x} = 1, \quad \int_{\mathbb{R}^3} f(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) d\mathbf{x} = f(\mathbf{y}). \quad (2.3.17)$$

Taking the Fourier transform of Eq. (2.3.15), we obtain,

$$-k^2 \hat{G} = e^{-i\mathbf{k} \cdot \mathbf{y}} \implies \hat{G} = \frac{e^{-i\mathbf{k} \cdot \mathbf{y}}}{k^2}, \quad (2.3.18)$$

where we have used the definition of the Fourier transform of a function $f(\mathbf{x})$ as,

$$\hat{f}(\mathbf{k}) = \int_{\mathbb{R}^3} f(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} d\mathbf{x}. \quad (2.3.19)$$

Simplifying $G(\mathbf{x}, \mathbf{y})$ using $\hat{G}$, we obtain (using the definition of the inverse Fourier transform),

$$G(\mathbf{x}, \mathbf{y}) = -\frac{1}{8\pi^3} \int_{\mathbb{R}^3} \frac{e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})}}{k^2} d\mathbf{k} \quad (2.3.20)$$

The integral can be evaluated using spherical coordinates. Taking the angle

of wavevector $\mathbf{k}$ with $\mathbf{x} - \mathbf{y}$ as θ , we can write

$$d\mathbf{k} = k^2 \sin \theta d\phi d\theta dk. \quad (2.3.21)$$

Upon substitution, we find,

$$\begin{aligned} G(\mathbf{x}, \mathbf{y}) &= -\frac{1}{4\pi^2} \int_0^\pi \int_0^\infty \frac{e^{ik|\mathbf{x}-\mathbf{y}|\cos\theta}}{k^2} k^2 dk d\sin\theta d\theta = \\ &= -\frac{1}{4\pi^2} \int_0^\infty \int_{-1}^1 e^{ik|\mathbf{x}-\mathbf{y}|t} dt dk = -\frac{1}{2\pi^2} \int_0^\infty \frac{\sin(kr)}{kr} dk, \end{aligned} \quad (2.3.22)$$

where $r = |\mathbf{x} - \mathbf{y}|$. The above integral can be simplified by considering dI/ds where $I(s) = \int_0^\infty e^{-sx} \frac{\sin x}{x} dx$, the details of which are left for the reader to outline. The final result of the integral yields the Green's function of the Poisson equation as,

$$G(\mathbf{x}, \mathbf{y}) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{y}|}. \quad (2.3.23)$$

Using the Green's function, the Poisson equation in 3D space can be solved as,

$$\nabla^2 \mathbf{f} = \mathbf{g} \text{ in } \mathbb{R}^3, \quad (2.3.24)$$

$$\Rightarrow \mathbf{f} = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\mathbf{f}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y}. \quad (2.3.25)$$

2.3.2 Line and sheet vortices

The above discussion motivates the definition of so-called line and sheet vortices. A line vortex is a generic curve about which vorticity can be assumed to be concentrated. Figure 2.4 shows a schematic illustration of a vortex curve C . The vorticity is assumed to be concentrated at the curve such that,

$$\int_{\Delta V} \boldsymbol{\omega} dV = K d\mathbf{l}, \quad (2.3.26)$$

where K is defined as the *strength of the vortex curve* and $d\mathbf{l}$ is the vector in the direction of the local vorticity. Hence, the induced velocity at a point $\mathbf{x}$ due to

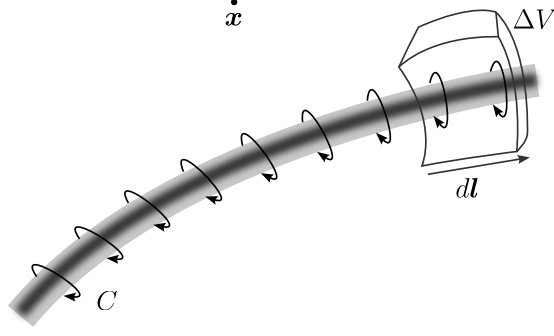


Figure 2.4: Illustration of a vortex curve C

the vortex curve is given by,

$$\mathbf{u}_v(\mathbf{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\boldsymbol{\omega} \times (\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^3} dV = -\frac{1}{4\pi} \int_C \frac{(\mathbf{x} - \mathbf{y}) \times K d\mathbf{l}}{|\mathbf{x} - \mathbf{y}|^3}. \quad (2.3.27)$$

Using this, it can be shown that velocity induced by a straight line vortex at a normal distance r from the line is $\frac{K}{2\pi r}$. A sheet vortex is defined as a collection



Figure 2.5: A sheet vortex, which is a collection of line vortices

of several line vortices. The strength of a sheet vortex is defined as,

$$\int_{\Delta V} \boldsymbol{\omega} dV = \int_S \boldsymbol{\Gamma} dS. \quad (2.3.28)$$

Using this, the induced velocity on either side of the sheet can be calculated as,

$$\mathbf{u}_v = -\frac{1}{4\pi} \int_S \frac{(\mathbf{x} - \mathbf{y}) \times \boldsymbol{\Gamma}}{|\mathbf{x} - \mathbf{y}|^3} dS. \quad (2.3.29)$$

For uniform strength (constant $\boldsymbol{\Gamma}$), we obtain,

$$\mathbf{u}_v = \frac{\boldsymbol{\Gamma}}{4\pi} \times \int_S \frac{(\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^3} dS = \frac{\boldsymbol{\Gamma}}{2} \times \hat{n}, \quad (2.3.30)$$

where $\hat{n}$ is the unit-normal passing through $\mathbf{x}$.

2.4 Reynolds Transport Theorem

Since only $\mathbf{S}$ consists of diagonal components of $\nabla \mathbf{u}$, the trace of $\mathbf{S}$ (S_{ii}) yields the rate of change of volume. This can be seen from the calculation of the rate of change of $\mathcal{J}$ as,

$$\frac{D\mathcal{J}}{Dt} = \frac{\partial \mathcal{J}}{\partial F_{iJ}} \frac{\partial F_{iJ}}{\partial t} \Big|_{\mathbf{x}} = \mathcal{J} \mathbf{F}^{-T} : \dot{\mathbf{F}} = \mathcal{J} [F^{-1}]_{Ji} \frac{\partial u_i}{\partial x_j} F_{jJ} = \mathcal{J} \frac{\partial u_i}{\partial x_j} \delta_{ji} = \mathcal{J} \frac{\partial u_i}{\partial x_i} = \mathcal{J} \nabla \cdot \mathbf{u}. \quad (2.4.1)$$

Using the above expression, a formal derivation of the Reynolds Transport Theorem (RTT) is possible (see [Aris \[2012\]](#) for example). Let $\varphi(\mathbf{x}, t)$ denote the amount of some quantity (any order tensor) per unit volume. Let B_0 denote the volume occupied by the deformable body in the reference configuration and B_t denote the volume occupied in the deformed configuration at time t . Then, according to RTT,

$$\frac{d}{dt} \int_{B_t} \varphi(\mathbf{x}, t) dV = \int_{B_t} \left(\frac{\partial \varphi}{\partial t} + \nabla \cdot (\varphi \mathbf{u}) \right) dV. \quad (2.4.2)$$

The proof of the above can be developed by changing the integral from deformed configuration to the reference configuration using $dV = \mathcal{J} dV_0$ as,

$$\int_{B_t} \varphi(\mathbf{x}, t) dV = \int_{B_0} \varphi(\mathbf{x}(\mathbf{X}, t), t) \mathcal{J} dV_0. \quad (2.4.3)$$

Now, since B_0 is not a function of time, the time derivative can be taken inside the integral to yield,

$$\frac{d}{dt} \int_{B_t} \varphi(\mathbf{x}, t) dV = \int_{B_0} \left(\frac{D\varphi}{Dt} \mathcal{J} + \varphi \frac{D\mathcal{J}}{Dt} \right) dV_0, \quad (2.4.4)$$

which simplifies as

$$\int_{B_0} \left(\frac{\partial \varphi}{\partial t} + \mathbf{u} \cdot \nabla \varphi + \varphi \nabla \cdot \mathbf{u} \right) \mathcal{J} dV_0, \quad (2.4.5)$$

which gives the R.H.S of the theorem statement Eq. 3.4.8 by converting back the integration domain from the reference configuration to the deformed configura-

tion,

$$\int_{B_t} \left(\frac{\partial \varphi}{\partial t} + \mathbf{u} \cdot \nabla \varphi + \varphi \nabla \cdot \mathbf{u} \right) dV = \int_{B_t} \left(\frac{\partial \varphi}{\partial t} + \nabla \cdot (\varphi \mathbf{u}) \right) dV. \quad (2.4.6)$$

It is important to note that no “hand-wavy” arguments were used in deriving this result. The derivation follows directly from the continuum mechanics tools, which are usually less appreciated in a more traditional fluid mechanics instructional environment. Even though less appreciated, the derivation should highlight the importance of looking at RTT through the lenses of continuum mechanics, rather than treating it as a purely fluid mechanics concept.

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