

Chapter 3

Governing equations and boundary conditions

In classical mechanics, conservation laws lay the foundation of mathematical modeling of any problem. These laws govern the evolution of some property of a material particle in time at every location in space. Derivation of such conservation laws uses mathematics to formalize physical concepts. These laws are fundamental in nature (axiomatic) and cannot be proven mathematically but are starting points for any mathematical exercise to simplify a problem.

In this chapter, we will first define some basic terminology using which we will derive the three independent dynamical (and thermodynamical) laws of continuum mechanics (applying both to solids and fluids): the mass conservation, momentum equation, and energy equation. There are additional laws/equations that may be derived using these conservation laws. We will look at kinetic energy equation, angular momentum equation, and the entropy equation as well. Then we will use simplified versions of these laws to look at special problems which are tractable analytically. As of now, a complete solution of these conservation equations is possible only numerically/computationally for general cases. While providing these laws, we also review some basic continuum thermodynamics.

3.1 System (control mass) and control volume

Beginning undergraduate students are often subjected to laws of motions that apply to a particle (Newton's laws of motion) or rigid bodies/collection of particles (Euler's axioms). Conservation laws in continuum mechanics generalize such laws

to any arbitrary bodies/volumes. A *system* of particles or just *system* is defined as an *arbitrary quantity of mass or collection of particles of fixed identity (same set/species of particles)*. A system often interacts with its *surroundings* via its *boundaries*. It is also referred as *control mass*, something which is identified by its own mass. On the other hand, a *control volume* is defined as a *region in space which might consist of some part of the system and its surroundings or both*, something which is identified by its volume only. It is a focused region in space, in which some *system* might enter or leave. A cinematically inclined reader might recall the beginning gun-barrel sequence of James Bond movies! The spy entering in to the focus constitutes the system, and the focused region of the gun-barrel is the control volume. In general, control volume V may move or deform, implying that both the control volume V and its control surface ∂V can change in time. Typical choice of control volume is to simplify the analysis of a problem. Any system can pass through the control surface since it is an imaginary surface in the space limiting our focus. Usually, it is advantageous to draw the control surface with a dashed line to remind oneself that any material can pass through this boundary.

3.2 Intensive and extensive quantities

The conservation laws in physics are defined for a *system*. For instance, Newton's second law states how the rate of change of linear momentum of a particle is proportional to the external force on that particle. Similarly, Euler's first axiom extends this to a system of particles stating that the sum of all external forces on a system is the rate of change of momentum of its center of mass. Conservation laws in continuum mechanics aim to extend these laws to deformable bodies. In fluid mechanics, we further aim to extend them to control volumes.

The motivation to this extension was presented in the previous chapter in which

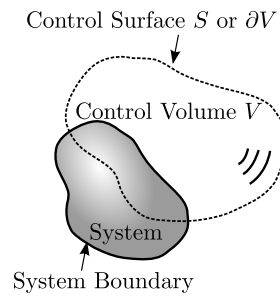


Figure 3.1: Control volume V , its surface S or ∂V , system and its boundary.

we saw that Eulerian fields are of importance in fluid mechanics. The conservation laws of the system govern the evolution of some *extensive* property Φ which pertains to the whole system. Mass density of such properties $\phi = d\Phi/dm$ are termed as *intensive* quantities. In thermodynamics, keyword *specific* is used for intensive quantities (specific energy, specific entropy, etc). Rigorously, intensive and extensive properties can be defined as follows, Consider a property, uniform in space, as a function of volume V , as $\Phi = \Phi(V)$. Here volume is the volume of the system. Now, if the volume of the system is scaled by a factor λ , i.e. $V \rightarrow \lambda V$, then say the property $\Phi \rightarrow \lambda^n \Phi$. If $n = 0$, then the property Φ is called an *intensive property* and if $n = 1$, then Φ is called an *extensive property*. In words, this implies that the extensive properties scale with the system's size, while intensive properties remain constant. In this fashion, density ρ can be categorized as an intensive property. However, if I keep scaling the volume down to particle level, then density is expected to change. How do we address this fallacy? Does it imply density is an extensive property? Also, what about properties which have $n \geq 2$?

For instance, if $\Phi = m$ is mass, then the corresponding intensive property is unity, if $\Phi = \mathbf{p} = m\mathbf{u}$ is linear momentum, then $\phi = \mathbf{u}$ is the velocity. Note that in above, mass and velocity correspond to the center of mass of the system. Correspondingly, the quantity $\rho\phi$, where $\rho = dm/dV$ is the density of the system, represents the density of the extensive property.

Additionally, the conservation laws of physics are defined when following the material points (Lagrangian). For instance,

$$\frac{d\mathbf{p}_{cm}}{dt} = \sum \mathbf{F}_{ext}, \quad (3.2.1)$$

where $\mathbf{p}_{cm}$ is the linear momentum of the center of mass of the system. Using Eq. (2.4.6), we can derive the final expression of the following laws applicable to

a system,

$$\text{Mass conservation : } \frac{d}{dt} \int_{B_t} \rho(\mathbf{x}, t) dV = 0, \quad (3.2.2)$$

$$\text{Euler's first axiom : } \frac{d}{dt} \int_{B_t} \rho(\mathbf{x}, t) \mathbf{u}(\mathbf{x}, t)_{P|I} dV \Big|_I = \mathbf{F}, \quad (3.2.3)$$

$$\text{Euler's second axiom : } \frac{d}{dt} \int_{B_t} \mathbf{x}_{PO} \times \mathbf{u}_{PO|I} \rho(\mathbf{x}_{PO}, t) dV \Big|_I = \mathbf{M}_O, \quad (3.2.4)$$

$$\text{First law of Thermodynamics : } \frac{dE}{dt} = \frac{\delta Q}{\delta t} + \frac{\delta W}{\delta t}. \quad (3.2.5)$$

The notation $\left(\frac{d}{dt} \right) \Big|_I$ denotes that the time derivative is computed w.r.t. to an inertial reference frame. In Eq. (3.2.4), the notation $_{PO}$ is used simply to reinforce that the point P 's position vector is evaluated w.r.t. O , a fixed point in the reference frame I . Note that in rigid body dynamics, this $\mathbf{x}_{PO}$ would be replaced by the material coordinates $\mathbf{X}_{PO}$. Since we are working with fields, we have used Eulerian coordinates and spatial fields in defining these laws/axioms.

Using the Reynolds Transport Theorem (derived in Sec 2.4), we obtain the following versions of the above laws.

$$\text{Mass equation : } \int_{B_t} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right) dV = 0, \quad (3.2.6)$$

$$\text{Momentum equation : } \int_{B_t} \left(\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) \right) dV = \mathbf{F}, \quad (3.2.7)$$

$$\text{Angular momentum equation : } \int_{B_t} \left(\frac{\partial (\rho \mathbf{x} \times \mathbf{u})}{\partial t} + \nabla \cdot (\rho (\mathbf{x} \times \mathbf{u}) \otimes \mathbf{u}) \right) dV = \mathbf{M}, \quad (3.2.8)$$

$$\text{Energy : } \int_{B_t} \left(\frac{\partial (\rho e)}{\partial t} + \nabla \cdot (\rho \mathbf{u} e) \right) dV = \frac{\delta Q}{\delta t} + \frac{\delta W}{\delta t}. \quad (3.2.9)$$

It is important to note that the limits of integration of above equations are over B_t , i.e., the deformed configuration. Hence, the above equations apply to deformed configuration composed of spatial points $\mathbf{x}$. Also, the above equations do not restrict how large deformed configuration one might consider. Consider movement of a deformable sponge (not fluid!), the above equations may apply to the whole sponge or very small sub-portion of the sponge. Applying the above equations

yields the governing equations of fluid mechanics in the differential form.

3.3 Mass equation or the continuity equation

Since the integral in Eq. (3.2.6) is zero for any arbitrary B_t , the integrand vanishes. Hence, the mass equation becomes,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (3.3.1)$$

which in Cartesian coordinates can be written using index notation as,

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} = 0. \quad (3.3.2)$$

Since the integral equation applies to B_t , which is a collection of spatial points, the differential equation applies to a given spatial point $\mathbf{x}$. The verbal reading of Eq. (3.3.1) is *The sum of rate of change of density at a point $\mathbf{x}$ and the divergence of momentum evaluated at the same point $\mathbf{x}$ is 0.*

Similar to the rate of change of the Jacobian in Eq. (2.4.1), the continuity equation relates velocity gradient and rate of change of volume (density) as,

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u} \implies \frac{1}{\rho} \frac{D\rho}{Dt} = -\nabla \cdot \mathbf{u}, \quad (3.3.3)$$

which is equivalent to Eq. (2.4.1) and highlights that the velocity divergence is equal to the negative of the rate of change of density per unit density.

3.3.1 Incompressibility

Throughout majority of this course, we will consider incompressible flows. Additionally, we will consider incompressible flows of a single species (no inhomogeneous mixtures). In such cases, the density of a fluid particle remains constant. This means, the material derivative of density vanishes,

$$\frac{D\rho}{Dt} = 0 \implies \nabla \cdot \mathbf{u} = 0. \quad (3.3.4)$$

Equation (3.3.4) can also be used in cases where density changes occur only due to temperature and the temperature changes are small. In such cases, we may consider that the ratio

$$\frac{\frac{D\rho}{Dt}}{\rho \nabla \cdot \mathbf{u}} \ll 1, \quad (3.3.5)$$

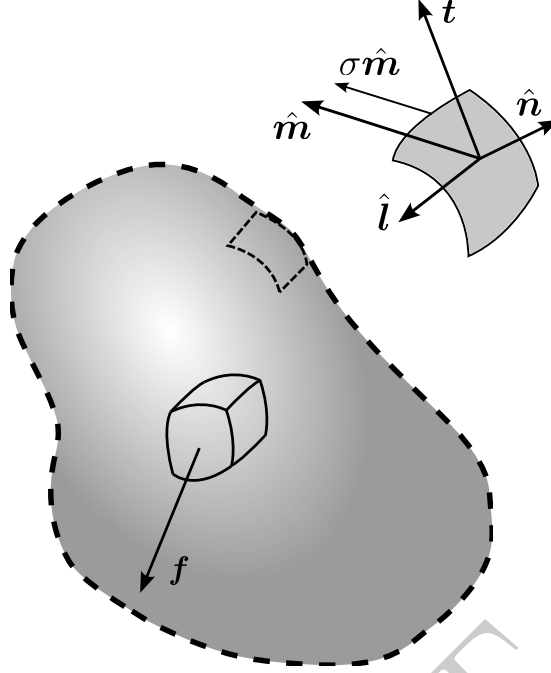


Figure 3.2: A control volume enclosing a blob of fluid in deformed configuration B_t , with surface traction force $\mathbf{t}$, volumetric force $\mathbf{f}$.

and use the incompressible continuity equation. This is a part of the so-called *Boussinesq approximation*, which will be considered in detail below. Such *non-isothermal* incompressible flows are useful for modeling natural convection phenomena.

3.4 Momentum equation

The momentum equation Eq. (3.2.7) can only be further simplified by considering the nature of all possible external forces. While the specific type of forces can be electrostatic, magnetic, gravitational, surface tension, etc, in general, the forces can be decomposed into *volumetric forces*, *surface tractions*, and *surface tension* forces. Hence, the external force can be written as,

$$\mathbf{F} = \int_{B_t} \mathbf{f} dV + \oint_{S_t} \mathbf{t} dS + \oint_{S_t} \sigma \hat{\mathbf{n}} dl. \quad (3.4.1)$$

For now, let us ignore the surface tension term. This is a valid assumption when the blob B_t exists in surroundings composed of the same fluid, or may be of a different fluid and $\sigma = 0$ is known/valid assumption. Substituting this in the

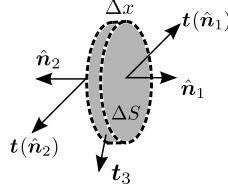


Figure 3.3: An arbitrary disc shaped control volume enclosing the system (fluid) of vanishing thickness with traction vectors on all its surfaces.

momentum equation, we obtain,

$$\int_{B_t} \left(\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) - \mathbf{f} \right) dV = \oint_{S_t} \mathbf{t} dS. \quad (3.4.2)$$

For further simplification, we need to understand the surface traction force $\mathbf{t}$ better, as discussed below.

3.4.1 Cauchy stress tensor

Consider the traction vectors $\mathbf{t}(\hat{\mathbf{n}}_1), \mathbf{t}(\hat{\mathbf{n}}_2), \mathbf{t}_3$ acting on an arbitrary disc type control volume shown in figure 3.3. Writing the momentum conservation law for this control volume (which coincides with the small elemental material region), and using the Gauss divergence theorem, we obtain (note that index notation is used for ease of the reader)

$$\left(\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} \right) \Delta S \Delta x = \rho f_i \Delta S \Delta x + t_i(\hat{\mathbf{n}}_1) \Delta S + t_i(\hat{\mathbf{n}}_2) \Delta S + t_{3i} \Delta x. \quad (3.4.3)$$

In the limit $\Delta x \rightarrow 0$, the contribution due to the volumetric terms approaches zero. Moreover, the contribution from the lateral surface also goes to zero. The remaining surface terms follow the relation,

$$t_i(\hat{\mathbf{n}}_1) \Delta S + t_i(\hat{\mathbf{n}}_2) \Delta S = 0. \quad (3.4.4)$$

Since the above equation holds for any arbitrary ΔS , then

$$t_i(\hat{\mathbf{n}}) = -t_i(-\hat{\mathbf{n}}). \quad (3.4.5)$$

The above relation states that the traction vector $\mathbf{t}$ is an odd function of the outward normal. It is a simple statement of Newton's third law according to which the reaction force from the fluid outside the control volume on the control

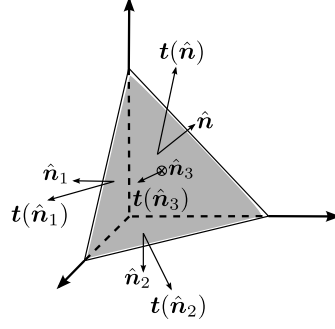


Figure 3.4: A small tetrahedral control volume enclosing a system (or a subsystem), with surface traction forces on all surfaces.

volume is equal and opposite to the force exerted by the fluid inside the control volume on the outside fluid (surroundings).

Transformation of traction vector with the surface orientation can tell us how the traction depends on the orientation (surface normal). To see how the traction vector transforms with the surface orientation, let us consider a tetrahedral control volume enclosing a system, as shown in figure 3.4. Applying the momentum balance as previously, we again see that in the limit of volume going to zero, surface forces balance out each other. Hence,

$$\mathbf{t}(\hat{\mathbf{n}})\Delta S + \mathbf{t}(\hat{\mathbf{n}}_1)\Delta S_1 + \mathbf{t}(\hat{\mathbf{n}}_2)\Delta S_2 + \mathbf{t}(\hat{\mathbf{n}}_3)\Delta S_3 = 0. \quad (3.4.6)$$

Given that $-\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}_i \Delta S = -n_i \Delta S = \Delta S_i$,

$$\mathbf{t}(\hat{\mathbf{n}}) = \mathbf{t}(\mathbf{n}_i)n_i = \mathbf{t}_i n_i = \mathbf{T} \cdot \mathbf{n}, \quad (3.4.7)$$

which in the index notation becomes,

$$t_j(n_i) = T_{ji}n_i \implies [\mathbf{T}] = \begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix} \quad (3.4.8)$$

In words, the above expression means that the traction vector transforms with the surface orientation, by being proportional to the surface normal. In general, proportionality between two vectors is via a second order tensor, which here is called the surface stress tensor $\mathbf{T}$ which is also called the Cauchy stress tensor.

Substituting this form of surface traction 3.4.7, we obtain,

$$\int_{B_t} \left(\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) - \mathbf{f} - \nabla \cdot \mathbf{T}^T \right) dV = 0. \quad (3.4.9)$$

It is important to note the $\nabla \cdot \mathbf{T}^T$ term. The stress tensor equation Eq. (3.4.8) could have been written as $t_j = n_i T_{ij}$ as well. In this case, the matrix representing the tensor $\mathbf{T}$ would consist of rows which are the traction vectors on respective faces. Consequently, the term in the momentum equation would be $\nabla \cdot \mathbf{T}$. The reader is encouraged to find that eventually, the terms included in the momentum equation are the same. Applying the same logic as the mass equation, we obtain the momentum equation as,

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = \mathbf{f} + \nabla \cdot \mathbf{T}^T, \quad (3.4.10)$$

which in index notation is,

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = f_i + \frac{\partial T_{ij}}{\partial x_j}. \quad (3.4.11)$$

An interesting result about the stress tensor $\mathbf{T}$ can be evaluated by considering the angular momentum equation.

3.4.2 Angular momentum equation

Since the net external forces on the blob are composed of $\mathbf{f}$ and $\mathbf{T} \cdot \hat{\mathbf{n}}$, the net external moment can be written as,

$$\mathbf{M}_O = \int_{B_t} \mathbf{x} \times \mathbf{f} dV + \oint_{S_t} \mathbf{x} \times (\mathbf{T} \cdot \hat{\mathbf{n}}) dS, \quad (3.4.12)$$

where $\mathbf{x}$ is the position vector w.r.t. O . Note that we have ignored any molecular level moments while writing $\mathbf{M}$. We assume that there are not such spin inducing moments acting on the fluid. Further, we will ignore any molecular level rotation of fluid particles as well, and thus write the following angular momentum equation,

$$\int_{B_t} \left(\frac{\partial(\rho \mathbf{x} \times \mathbf{u})}{\partial t} + \nabla \cdot (\rho (\mathbf{x} \times \mathbf{u}) \mathbf{u}) - \mathbf{x} \times \mathbf{f} - \nabla \cdot (\mathbf{x} \times \mathbf{T}^T) \right) dV = 0, \quad (3.4.13)$$

which in differential form and index notation becomes,

$$\frac{\partial(\epsilon_{ijk} \rho x_j u_k)}{\partial t} + \frac{\partial(\rho u_l \epsilon_{ijk} x_j u_k)}{\partial x_l} = \epsilon_{ijk} x_j f_k + \frac{\partial(\epsilon_{ijk} x_j T_{kl})}{\partial x_l}. \quad (3.4.14)$$

Now, taking the cross product of $\mathbf{x}$ with Eq. (3.4.11) we obtain,

$$\epsilon_{ijk}x_j \frac{\partial(\rho u_k)}{\partial t} + \epsilon_{ijk}x_j \frac{\partial(\rho u_k u_l)}{\partial x_l} = \epsilon_{ijk}x_j f_k + \epsilon_{ijk}x_j \frac{\partial T_{kl}}{\partial x_l}. \quad (3.4.15)$$

Subtracting from Eq. (3.4.14), we obtain (noting that $\epsilon_{ijk}u_j u_k = 0$),

$$\epsilon_{ijk}\rho u_k u_l \frac{\partial x_j}{\partial x_l} = \epsilon_{ijk}T_{il} \frac{\partial x_j}{\partial x_l} \implies \epsilon_{ijk}T_{ij} = 0. \quad (3.4.16)$$

This implies $T_{ij} = T_{ji}$ or $\mathbf{T}^T = \mathbf{T}$. The stress tensor is symmetric.

3.4.3 Newtonian fluids and Navier-Stokes equations

Further simplification of the differential momentum conservation eq. ?? depends on further simplification of the stress tensor $\mathbf{T}$. *Newtonian fluids* are those fluids for which the stress tensor $\mathbf{T}$ can be written as proportional to the local velocity gradient $\nabla \mathbf{u}$, i.e. in index notation,

$$T_{ij} = C_{ijkl} \frac{\partial u_k}{\partial x_l} + C \delta_{ij}. \quad (3.4.17)$$

Imposing the symmetric condition on the stress tensor derived previously, and assuming isotropic nature of the fluid, the fourth order tensor C_{ijkl} maybe written in general as,

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \quad (3.4.18)$$

Using the above general expression of a symmetric fourth order isotropic tensor, the stress tensor $\mathbf{T}$ can be written as,

$$\mathbf{T} = -p\mathbf{I} + \lambda \nabla \cdot \mathbf{u} \mathbf{I} + 2\mu \left(\frac{\nabla \mathbf{u} + \nabla \mathbf{u}^T}{2} \right) = -p\mathbf{I} + \boldsymbol{\sigma}, \quad (3.4.19)$$

where we have replaced C with $-p$ and coefficients 2μ and λ are often referred as the first and second viscosity coefficients. This p can be seen as the normal stress which would act in all three directions, if the fluid is at rest (not flowing). We know, that such stress is due to the thermodynamic pressure, which in term is due to the vibration of molecules of the fluid. The remaining part of the stress tensor is often called the *deviatoric* stress tensor $\boldsymbol{\sigma}$. We can define a *mechanical pressure* p_m as *average normal stresses in all directions*, i.e.,

$$p_m = -\frac{T_{ii}}{3}. \quad (3.4.20)$$

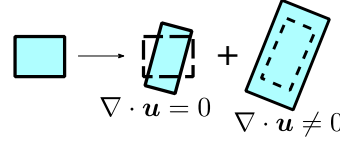


Figure 3.5: General decomposition of deformation of a fluid element

The thermodynamic pressure p and the mechanical pressure p_m are related as,

$$p_m = p - \left(\lambda + \frac{2\mu}{3} \right) \nabla \cdot \mathbf{u} = p - \kappa \nabla \cdot \mathbf{u}, \quad (3.4.21)$$

where κ is identified as the *coefficient of bulk viscosity*. Such bulk viscous stresses occur only in compressible flows where dilatation of a material element takes place, and maybe visualized mechanically as the dampers which slow down only volumetric expansion and contraction of a fluid element (see figure 3.5). Substituting the above definition of the bulk viscosity κ , the Navier-Stokes equations for a fully compressible fluid become,

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla p + \rho \mathbf{f} + \nabla \cdot \left(\mu (\nabla \mathbf{u} + \nabla \mathbf{u}^T) + \left(\kappa - \frac{2\mu}{3} \right) (\nabla \cdot \mathbf{u}) \mathbf{I} \right). \quad (3.4.22)$$

For simplicity, the above equations are sometimes condensed as,

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla p + \rho \mathbf{f} + \nabla \cdot \left(\mu \mathbf{S} + \left(\kappa - \frac{2\mu}{3} \right) \mathbf{D} \right), \quad (3.4.23)$$

where $\mathbf{S}$ and $\mathbf{D}$ are strain rate and dilatation rate (written as dilatation only) tensors, respectively. Multiplying the continuity equation with $\mathbf{u}$, the above equation can be simplified further as,

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mathbf{f} + \nabla \cdot (\mu \mathbf{S}) + \nabla \cdot \left(\kappa - \frac{2\mu}{3} \right) \mathbf{D}. \quad (3.4.24)$$

Throughout this course, we are going to assume flows of fluids with constant viscosity. Furthermore, we are going to assume $\nabla \cdot \mathbf{u} = 0$ to hold for most of the course, which yields the simplified Navier-Stokes equation for incompressible flows,

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{f} + \mu \nabla^2 \mathbf{u}. \quad (3.4.25)$$

3.4.4 Vorticity equation

Vorticity $\boldsymbol{\omega}$ is defined as $\nabla \times \mathbf{u}$ and quantifies the local rate of rotation of a fluid element (see Section 2.3). A transport equation for vorticity can be derived by taking the curl of the momentum equation such that,

$$\nabla \times \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\rho} = \frac{1}{\rho} \nabla \cdot \boldsymbol{\sigma} \right), \quad (3.4.26)$$

which yields,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = \frac{\nabla p \times \nabla \rho}{\rho^2} + \nabla \left(\frac{\nabla \cdot \boldsymbol{\sigma}}{\rho} \right). \quad (3.4.27)$$

The first term on the right hand side is called the *baroclinic moment* which results in generation of vorticity in compressible flows. For incompressible flows (to be specific constant ρ or $\nabla \cdot \mathbf{u} = 0$), the equation simplifies to,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = \nu \nabla \times \nabla^2 \mathbf{u} = \nu \nabla^2 \boldsymbol{\omega}. \quad (3.4.28)$$

Simplifying $\nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = \mathbf{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \mathbf{u}$ (show this), we obtain,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}. \quad (3.4.29)$$

The term $\nabla^2 \boldsymbol{\omega}$ represents simply the diffusion of vorticity. The term $\boldsymbol{\omega} \cdot \nabla \mathbf{u}$ represents the rate of change of vorticity due to *stretching and tumbling of vortex lines*. Essentially, $\boldsymbol{\omega} \cdot \nabla \mathbf{u} = \boldsymbol{\omega} \cdot (\mathbf{S} + \boldsymbol{\Omega})$. For a fluid with constant density, both pressure forces and gravitational forces will pass through the center of mass of a fluid element, and hence do not generate any moment. Thus they do not appear in the vorticity equation.

3.4.5 Surface tension force

Surface tension force occurs at an interface between two fluids. It is a result of differential in forces between molecules on either side of the interface. As a result of which, the surface (interface) requires an energy for creation and can not spontaneously occur in a homogeneous mixture. Figure 3.6 shows the schematic of an interface between two fluids *A* and *B*. Consider an element of the interface shown zoomed in. The normal to the element is $\hat{\mathbf{n}}$. $\hat{\mathbf{l}}$ runs through the curved edge of the interface and $\hat{\mathbf{m}}$ is the direction normal to $\hat{\mathbf{l}}$ and $\hat{\mathbf{n}}$, along which the

surface tension force occurs ($\hat{\mathbf{m}} = \hat{\mathbf{l}} \times \hat{\mathbf{n}}$). The surface tension force on a fluid

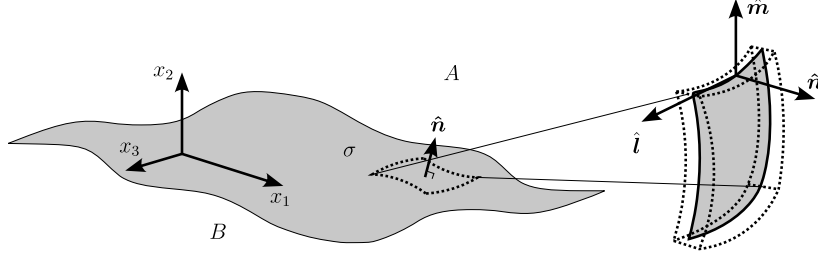


Figure 3.6: Interface between two fluids with surface tension force stretching an element on the interface along the $\hat{\mathbf{m}}$ direction.

element sandwiching the interface is

$$\oint \sigma \hat{\mathbf{m}} dl, \quad (3.4.30)$$

where σ is the coefficient of surface tension, and is a property of *both* the fluids A and B . To simplify recall the Stokes' theorem,

$$\oint \hat{\mathbf{l}} \times \mathbf{f} dl = \hat{\mathbf{e}}_i \oint \epsilon_{ijk} l_j f_k dl = \hat{\mathbf{e}}_i \int \epsilon_{ijk} n_j \epsilon_{klm} \frac{\partial f_m}{\partial x_l} dS = \int (\hat{\mathbf{n}} \times \nabla) \times \mathbf{f} dS. \quad (3.4.31)$$

Considering $\mathbf{f} = \sigma \hat{\mathbf{n}}$, we get,

$$\oint \sigma \hat{\mathbf{m}} dl = \oint \hat{\mathbf{l}} \times \sigma \hat{\mathbf{n}} dl = \int (\hat{\mathbf{n}} \times \nabla) \times \sigma \hat{\mathbf{n}} dS. \quad (3.4.32)$$

Before proceeding, we define a *surface gradient operator* ∇_S as,

$$\nabla_s = \nabla - \hat{\mathbf{n}} (\hat{\mathbf{n}} \cdot \nabla), \quad (3.4.33)$$

where we remove the normal component of the gradient to obtain variation over the surface. Furthermore, we note that

$$\hat{\mathbf{n}} \times \nabla = \hat{\mathbf{n}} \times \nabla_S. \quad (3.4.34)$$

Now, simplifying the term in Eq. (3.4.32), we obtain,

$$\oint \sigma \hat{\mathbf{m}} dl = \int (\nabla_S \sigma - \hat{\mathbf{n}} \sigma \nabla_S \cdot \hat{\mathbf{n}}) dS. \quad (3.4.35)$$

By definition, *mean curvature* $\mathcal{H}$ in 3D is defined as

$$\mathcal{H} = -\frac{1}{2} \nabla \cdot \hat{\mathbf{n}} = -\frac{1}{2} \nabla_S \cdot \hat{\mathbf{n}}, \quad (3.4.36)$$

which yields the formula,

$$\oint \sigma \hat{\mathbf{n}} dl = \int (\nabla_S \sigma + 2\mathcal{H}\sigma \hat{\mathbf{n}}) dS. \quad (3.4.37)$$

Since, this integral has to be performed over the whole closed area, we may replace $\int$ with $\oint_{S_t}$ to obtain the expression of the surface tension force,

$$\oint_{S_t} (\nabla_S \sigma + 2\mathcal{H}\sigma \hat{\mathbf{n}}) dS. \quad (3.4.38)$$

The mean curvature is positive if the surface curves towards the normal. Shown in figure 3.6, the surface element bends away from the normal, hence $\mathcal{H}$ is negative. Consequently, the surface tension force is opposite to the normal $\hat{\mathbf{n}}$ (considering uniform $\nabla_S \sigma$). Note that the force combines with pressure in the overall momentum equation. As seen in subsequent sections, the surface tension is used in deriving the so-called *interface conditions*.

3.5 Energy equation

Before we begin our discussion of the total energy equation (or what total energy is), we can use Eq. (3.4.25) to derive the transport equation for kinetic energy per unit volume, defined as $\rho|\mathbf{u}|^2/2$ or simply $\rho u^2/2$ by taking the dot product of Eq. (3.4.25) with $\mathbf{u}$,

$$\rho \left(\frac{\partial}{\partial t} \left(\frac{u^2}{2} \right) + \mathbf{u} \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) \right) = -\mathbf{u} \cdot \nabla p + \mathbf{u} \cdot \mathbf{f} + \mu \mathbf{u} \cdot \nabla^2 \mathbf{u}. \quad (3.5.1)$$

The second term can be further simplified as,

$$\mathbf{u} \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) = u_i u_j \frac{\partial u_i}{\partial x_j} = u_j \frac{\partial}{\partial x_j} \left(\frac{u_i u_i}{2} \right) = \mathbf{u} \cdot \nabla \left(\frac{u^2}{2} \right), \quad (3.5.2)$$

which yields,

$$\rho \left(\frac{\partial k}{\partial t} + \mathbf{u} \cdot \nabla k \right) = -\mathbf{u} \cdot \nabla p + \mathbf{u} \cdot \mathbf{f} + \mu \mathbf{u} \cdot \nabla^2 \mathbf{u}, \quad (3.5.3)$$

where $k = u^2/2$ is the kinetic energy per unit mass.

3.5.1 Total energy equation

In general, a fluid will consist of both kinetic and internal energy. The potential energy is accounted for by the $\mathbf{u} \cdot \mathbf{f}$ term in which the force $\mathbf{f}$ consists of all the potential as well as non-potential (non conservative) forces. The total energy of a fluid blob at temperature T , given by E is,

$$E = \int_{B_t} \rho \int_0^T c_v dT dV + \int_{B_t} \rho k dV, \quad (3.5.4)$$

where the first term denotes the internal energy of the fluid and c_v is the specific heat capacity at constant volume. See detailed discussion of compressible flows for a topical review of classical thermodynamics. The total specific energy is given by,

$$e = \rho \int_0^T c_v dT + k. \quad (3.5.5)$$

From first law of thermodynamics, the total rate of change of the total energy is equal to the total heat supplied to the system and the total *work done on the system*,

$$\frac{dE}{dt} = \frac{\delta Q}{\delta t} + \frac{\delta W}{\delta t}, \quad (3.5.6)$$

where δ is used instead of total derivative to highlight that these are path dependent (not complete derivatives) quantities. Using the Reynolds transport theorem, we obtain,

$$\int_{B_t} \left(\frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho \mathbf{u} e) \right) dV = \frac{\delta Q}{\delta t} + \frac{\delta W}{\delta t}. \quad (3.5.7)$$

The total rate of work done consists of the work done by both volumetric as well as surface forces,

$$\frac{\delta W}{\delta t} = \int_{B_t} \mathbf{f} \cdot \mathbf{u} dV + \oint_{S_t} \mathbf{T} \cdot \hat{\mathbf{n}} \cdot \mathbf{u} dS. \quad (3.5.8)$$

The total rate of heat supplied consists of any volumetric heating which may occur within the body and any heat supplied through the boundaries. This requires definition of a *heat flux* $\mathbf{q}$ such that,

$$\frac{\delta Q}{\delta t} = \int_{B_t} \dot{S} dV - \oint_{S_t} \mathbf{q} \cdot \hat{\mathbf{n}} dS. \quad (3.5.9)$$

Which yields, upon using divergence theorem,

$$\frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho \mathbf{u} e) = \mathbf{f} \cdot \mathbf{u} + \nabla \cdot (\mathbf{T} \cdot \mathbf{u}) - \nabla \cdot \mathbf{q} + \dot{S}. \quad (3.5.10)$$

A general version of the kinetic energy equation is obtained through scalar product of velocity field with the momentum equation 3.4.10 ,

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho |\mathbf{u}|^2 \right) + \nabla \cdot \left(\mathbf{u} \frac{1}{2} \rho |\mathbf{u}|^2 \right) = \mathbf{u} \cdot \nabla \cdot \mathbf{T} + \mathbf{f} \cdot \mathbf{u}, \quad (3.5.11)$$

where $\mathbf{u} \cdot \nabla \cdot \mathbf{T}$ accounts for the rate of work done on a fluid element as a result of the force imbalance on the element and its velocity. However, this is only mechanical work done on the element and not the total work done on the element due to the stress tensor $\mathbf{T}$. The remaining part of the work is done in *deforming* the fluid element as a result of it's deformation rate quantified by $\nabla \mathbf{u}$. Thus, the total work done on the fluid element is,

$$\nabla \cdot (\mathbf{T} \cdot \mathbf{u}), \quad (3.5.12)$$

which integrated over a small element yields, $\int_{S_t} (\mathbf{T} \cdot \mathbf{u}) \cdot \hat{\mathbf{n}} dS$. Note that the fluid deformation rate $\nabla \mathbf{u}$ also consists of the rotation rate, the anti-symmetric part, and the deformation or *strain rate*, the symmetric part. Moreover, the stress tensor $\mathbf{T}_{ij}$ is symmetric, as a result of which, the deformation work can be written as ,

$$\mathbf{T} : \nabla \mathbf{u} = T_{ij} \frac{\partial u_i}{\partial x_j} = \mathbf{T} : \mathbf{S}, \quad (3.5.13)$$

where $\mathbf{S}$ is the strain rate. Substituting the newtonian assumption for the fluid viscosity, the deformation work becomes,

$$\mathbf{T} : \mathbf{S} = -p \nabla \cdot \mathbf{u} + 2\mu \mathbf{S} : \mathbf{S} + \left(\kappa - \frac{2\mu}{3} \right) (\nabla \cdot \mathbf{u})^2, \quad (3.5.14)$$

which can be written as ,

$$\mathbf{T} : \mathbf{S} = -p \nabla \cdot \mathbf{u} + \phi, \quad (3.5.15)$$

where ϕ is the viscous dissipation of energy. For vanishing bulk viscosity $\kappa = 0$, the viscous dissipation can be written as,

$$\phi = 2\mu \left(S_{ij} S_{ij} - \frac{1}{3} (\nabla \cdot \mathbf{u})^2 \right) = 2\mu \left(S_{ij} - \frac{1}{3} (\nabla \cdot \mathbf{u}) \delta_{ij} \right)^2 \geq 0. \quad (3.5.16)$$

Subtracting Eq. (3.5.11) from the total energy equation, we obtain,

$$\frac{\partial(\rho e_I)}{\partial t} + \nabla \cdot (\rho \mathbf{u} e_I) = -\nabla \cdot \mathbf{q} + \dot{S} - p \nabla \cdot \mathbf{u} + \phi, \quad (3.5.17)$$

which highlights that the viscous dissipation term, which is always non-negative, results in a non-negative rise in temperature as a result of dissipation. Such dissipation phenomena are at the core of non equilibrium thermodynamics and result in an increasing entropy of an isolated system. However, along with the viscous dissipation, thermal dissipation or *diffusion* is also present in fluids. In a fluid element, the heat transfer occurs due to thermal transport from nearby fluid surroundings. Such heat transfer is usually modelled using the Fourier law, according to which,

$$\mathbf{q} = -k \nabla T, \quad (3.5.18)$$

where k is thermal conductivity.

3.6 Navier-Stokes in Rotating frame

So far, we have derived the Navier-Stokes equations in an inertial frame of reference. Note that, the spatial coordinates $\mathbf{x}$ chosen in all derivations, correspond to the coordinates of the material points in the deformed configuration. Throughout, we chose to represent $\mathbf{x}$ using unit vectors which were fixed in time. However, most of the fluids that we observe, are on Earth, which is rotating and speeding through a galaxy. Observation of fluid flows on different planets and stars would require their relative motion to be accounted for. To this end, we recall the following. Consider a material point coordinate $\mathbf{x}$, specified in a frame which itself is accelerating. Hence,

$$\mathbf{x} = x_i \hat{\mathbf{e}}_i(t). \quad (3.6.1)$$

Consequently, the spatial velocity field of the material points, obtained as,

$$\left. \frac{\partial \mathbf{x}}{\partial t} \right|_{\mathbf{x}}. \quad (3.6.2)$$

It is important to note that to apply Euler's axioms, we required velocity and acceleration in an inertial reference frame, which a generally rotating frame is not. Hence, the time derivative must be taken in an inertial reference frame

which yields,

$$\left. \frac{\partial \mathbf{x}}{\partial t} \right|_{\mathbf{x}, I} = \frac{\partial x_i}{\partial t} \hat{\mathbf{e}}_i + x_i \frac{\partial \mathbf{e}_i}{\partial t}, \quad (3.6.3)$$

where the additional I in subscript indicates time derivative in an inertial frame.

Which yields,

$$\mathbf{u}_I = \mathbf{u}_R + \boldsymbol{\Omega} \times \mathbf{x}. \quad (3.6.4)$$

Similarly, acceleration in inertial reference frame becomes,

$$\mathbf{a}_I = \mathbf{a}_R + \left. \frac{d\mathbf{u}_R}{dt} \right|_R + \dot{\boldsymbol{\Omega}} \times \mathbf{x} + 2\boldsymbol{\Omega} \times \mathbf{u}_R + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}), \quad (3.6.5)$$

where $\boldsymbol{\Omega}$ and $\dot{\boldsymbol{\Omega}}$ are the rotation rates and angular acceleration of the frame, respectively. $\mathbf{a}_R$ is the translational acceleration of the frame. Before substituting in the RTT, we must recall that the fluid flow is being measured from a rigid accelerating frame. Hence, the velocity $\mathbf{u}_R$ is the velocity of the fluid. Since the Euler's axiom only applies in an inertial frame, we have to modify Eq. (3.2.7) such that,

$$\frac{d}{dt} \int_{B_t} (\rho \mathbf{u} + \rho \boldsymbol{\Omega} \times \mathbf{x}) dV = \mathbf{F}, \quad (3.6.6)$$

which yields the following equation in the differential form

$$\rho \frac{D\mathbf{u}}{Dt} = \nabla \cdot \mathbf{T} + \mathbf{f} - 2\rho \boldsymbol{\Omega} \times \mathbf{u} - \rho \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}) - \rho \mathbf{a}_R. \quad (3.6.7)$$

3.7 Boussinesq approximation and natural convection

Before discussing Boussinesq approximation, we address the concept of incompressibility in more detail. To be exact, compressibility of a fluid is the fact that density of a fluid element changes due to pressure changes around the fluid element. The incompressibility assumption made earlier (density of a fluid element is constant) can be refined further by saying that *the density of a fluid element is independent of the pressure*. Temperature changes may cause density changes, and the flow may still be called incompressible. Convection of water heating up in a pot is an example of this. In atmosphere and oceans, convection of air/water is an example. One must be careful though. Over very large lengths in either, the density changes due to pressure become important and the incompressibility

may not be perfectly valid. Assuming density changes are coupled only with temperature, we write,

$$\frac{\delta\rho}{\rho} = -\alpha\delta T, \quad (3.7.1)$$

where α is the so-called *thermal expansion coefficient*.

Within Boussinesq approximation, we say that δT is so small, that $\delta\rho/\rho$ is small. Consequently, the continuity equation may be written as,

$$\frac{\epsilon}{\rho} \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0, \quad (3.7.2)$$

where $\epsilon \ll 1$. How this ϵ appears requires us to understand non-dimensionalization, which we will consider in the next chapter, where we will give full derivation of the Boussinesq approximation. Here, we will just obtain the final result for completeness.

For very small δT , ignoring the density changes in the continuity equation, we retain

$$\nabla \cdot \mathbf{u} = 0. \quad (3.7.3)$$

In the momentum equation (assuming Newtonian isotropic fluid), we use the simplified continuity equation to obtain,

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{f} + \mu \nabla^2 \mathbf{u}, \quad (3.7.4)$$

where $\mathbf{f} = \rho\mathbf{g}$ for gravitational forces. Let the density of the fluid be ρ_0 when the temperature is uniform. When fluid is at rest in gravity, pressure will exhibit hydrostatic variation. Let this pressure be called the *base pressure* p_0 . It is governed by

$$\nabla p_0 = \rho_0 \mathbf{g}. \quad (3.7.5)$$

Let there be small temperature variations, as a result of which density deviates from the base value by a small amount $\rho = \rho_0 + \rho'$. As a result of temperature variations, the fluid will no longer be at rest and exhibit some velocity $\mathbf{u}'$. Consequently,

$$\left(1 + \frac{\rho'}{\rho_0}\right) \frac{D\mathbf{u}'}{Dt} = -\frac{\nabla p'}{\rho_0} + \frac{\rho'}{\rho_0} \mathbf{g} + \nu \nabla^2 \mathbf{u}', \quad (3.7.6)$$

where we have divided the equation by ρ_0 . Ignoring the product of small terms in

on the LHS, we obtain the momentum equation under Boussinesq approximation,

$$\frac{D\mathbf{u}'}{Dt} = -\frac{\nabla p'}{\rho_0} + \frac{\rho'}{\rho_0}\mathbf{g} + \nu\nabla^2\mathbf{u}'. \quad (3.7.7)$$

In Eq. (3.7.7), the density variations play a role in gravitational forces (bouyancy) which may drive the flow. However, density variations do not appear in any other term in the equation.

The internal energy Eq. (3.5.17) can be further simplified as well. Assuming density changes are only due to temperature changes, we write,

$$p\nabla \cdot \mathbf{u} = -\frac{p}{\rho} \frac{D\rho}{Dt} = -\frac{p}{\rho} \frac{d\rho}{dT} \frac{DT}{Dt} = p\alpha \frac{DT}{Dt}. \quad (3.7.8)$$

For ideal gas, $\alpha = 1/T$, which yields,

$$p\nabla \cdot \mathbf{u} = \rho R \frac{DT}{Dt} = \rho (c_p - c_v) \frac{DT}{Dt}. \quad (3.7.9)$$

Substituting the above in the internal energy equation, writing $e_I = c_v T$, we obtain,

$$\rho c_p \frac{DT}{Dt} = -\nabla \cdot \mathbf{q} + \phi. \quad (3.7.10)$$

In liquids, $\alpha \ll 1/T$, as a result of which, the $p\nabla \cdot \mathbf{u}$ term can be ignored in comparison to other dominant terms, yielding the temperature equation under Boussinesq approximation as,

$$\rho c_v \frac{DT}{Dt} = -\nabla \cdot \mathbf{q} + \phi. \quad (3.7.11)$$

3.8 Boundary and interface conditions

To determine fluid flow, equations (3.3.1), (3.4.10), and (3.5.10) have to be solved. These are nonlinear partial differential equations in variables $\rho, \mathbf{u}, p, T$. Since total unknowns are 6, one additional equation is required. For compressible flows, this equation is supplied by the relation between p, ρ , and T (equation of state).

For an isothermal incompressible flow, the governing equations to be solved are,

$$\nabla \cdot \mathbf{u} = 0, \quad (3.8.1)$$

$$\frac{D\mathbf{u}}{Dt} = -\frac{\nabla p}{\rho} + \nu\nabla^2\mathbf{u}. \quad (3.8.2)$$

These are 4 equations in 4 variables p and $\mathbf{u}$. The continuity equation is a first order PDE in all directions. The momentum equation is a second order PDE in each spatial direction and first order PDE in time. Hence, to solve the system in a domain (bounded or unbounded), two boundary conditions are required in each direction and one initial condition is required in time.

3.8.1 Hard wall

A hard wall is a boundary of a fluid domain at which the fluid has velocity equal to that of the wall in all directions. This boundary condition is a combination of two boundary conditions, namely: no-slip boundary and no-penetration boundary. Consider an arbitrary wall with local normal $\hat{\mathbf{n}}$ as shown in figure 3.7. At any point on the hard wall, we have

$$(\mathbf{u} - \hat{\mathbf{n}}(\mathbf{u} \cdot \hat{\mathbf{n}})) \Big|_{wall} = (\mathbf{v}_w - \hat{\mathbf{n}}(\mathbf{v}_w \cdot \hat{\mathbf{n}})) \quad (\text{no slip}), \quad (3.8.3)$$

and

$$\mathbf{u} \cdot \hat{\mathbf{n}} \Big|_{wall} = \mathbf{v}_w \cdot \hat{\mathbf{n}} \quad (\text{no penetration}), \quad (3.8.4)$$

where $\mathbf{v}_w$ is the velocity of the wall.

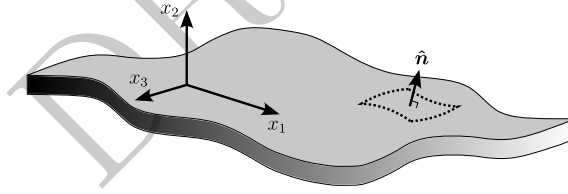


Figure 3.7: Wall with an arbitrary shape in 3D.

3.8.2 Porous wall

For a porous wall, the no slip boundary condition can be combined with the flow driven by pressure difference across the wall. For pressures p_1 and p_2 on either side of the wall, the normal velocity is given by

$$\mathbf{u} \cdot \hat{\mathbf{n}} \Big|_{wall} = \frac{\kappa}{\mu} (p_1 - p_2) \quad (\text{porous wall}), \quad (3.8.5)$$

where κ denotes the *permeability* of the wall and μ is the dynamic viscosity of the fluid.

3.8.3 Infinity

Often, in fluid dynamics, it is informative to solve a canonical problem representative of the real world application. The canonical problem is crafted or chosen such that the essential/required physics in the real world application is retained, without the “noise” of other phenomena. For instance, boundary layer formation over an infinite flat plate might hardly be seen in a real world application. However, insights from this canonical problem have been used for all aerodynamic design.

In an effort to isolate the physics, often (but not necessarily) the domain of a canonical problem is unbounded in atleast one direction. That is, the flow quantities are expected to show some asymptotic behavior, as we approach ∞ in that direction. Boundary conditions in such case are usually called the *infinity* boundary condition.

The exact mathematical expression of the infinity boundary condition depends on the problem. But the thumb rule is: *the kinematic effect of some external surface/body/perturbation takes a very long time to reach infinity, hence, at infinity, the flow can be assumed to not be impacted by the geometry/perturbation in the problem.*

3.8.4 Interface condition

The interface condition at the interface between two fluids is also a two part condition. The first part is the *kinematic interface condition* and the second part is the *dynamic interface condition*.

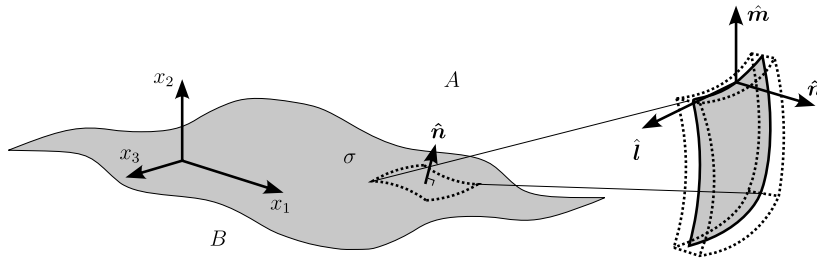


Figure 3.8: An interface between two flow streams in x direction. A zoomed-in section shows the local surface coordinate system used for considering the surface tension force.

Kinematic interface condition

There is a great deal of confusion regarding the kinematic interface condition. Let us first try the following basic procedure, which is shown in a lot of fluid mechanics lectures and see what the result yields. First, recall that for a moving control volume, the mass equation obtained reads,

$$\frac{d}{dt} \int_{V_t} \rho dV = - \oint_{S_t} \rho (\mathbf{u} - \mathbf{w}) \cdot \hat{\mathbf{n}} dS, \quad (3.8.6)$$

where V_t is the changing control volume boundary, S_t is the time changing surface of the control volume, and $\mathbf{w}$ is the velocity of the control volume at the surface point. Now consider a small control volume moving with the interface as shown in figure 3.9. As the interface deforms, the control volume moves with it. Let us apply Eq. (3.8.6) in the limit $\delta l \rightarrow 0$ such that $V_t \rightarrow 0$. Since the control volume moves with the interface, the mass inside the CV is fixed, hence the LHS approaches 0 on account of vanishing volume. For RHS, we note that the control volume is moving with interface. Hence, the relative velocity of the fluid at the top and bottom of the fluid must be 0! Interestingly, the equation yields,

$$\rho (\mathbf{u} - \mathbf{w}) \cdot \hat{\mathbf{n}} S \Big|_A = \rho (\mathbf{u} - \mathbf{w}) \cdot \hat{\mathbf{n}} S \Big|_B = 0. \quad (3.8.7)$$

Hence, applying the equation to a moving CV does not yield anything useful for kinematic interface condition. The conceptual statement that the CV velocity and the fluid velocity on both the sides of the interface is the statement of the kinematic interface condition. To get the equation, we use the equation of the interface,

$$f = x_2 - \xi(x_1, x_3, t) = \text{const..} \quad (3.8.8)$$

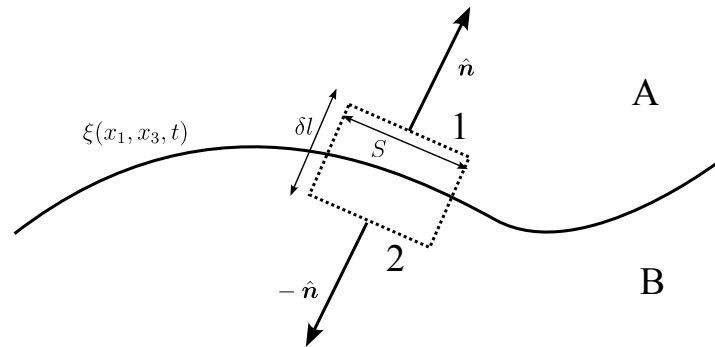


Figure 3.9: A small control volume moving with the interface.

Taking the material derivative, we obtain,

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f = 0 \implies \mathbf{u} \cdot \nabla f = \frac{\partial \xi}{\partial t}, \quad (3.8.9)$$

or

$$\mathbf{u} \cdot \hat{\mathbf{n}} = \frac{1}{|\nabla f|} \frac{\partial \xi}{\partial t} \text{ at the interface.} \quad (3.8.10)$$

Hence, the normal velocity is equal on both sides of the interface, and is equal to $\frac{1}{|\nabla f|} \frac{\partial \xi}{\partial t}$. Furthermore, since the velocities are equal and their normal components are equal, we obtain that *no slipping* takes place at the interface, i.e.,

$$(\mathbf{u} - \hat{\mathbf{n}}(\mathbf{u} \cdot \hat{\mathbf{n}})) \Big|_1 = (\mathbf{u} - \hat{\mathbf{n}}(\mathbf{u} \cdot \hat{\mathbf{n}})) \Big|_2 \text{ at the interface.} \quad (3.8.11)$$

Dynamic interface condition

To obtain the dynamic interface condition, we recall the momentum equation applied to the deforming control volume,

$$\frac{d}{dt} \int_{V_t} \rho \mathbf{u} dV = - \oint_{S_t} \rho \mathbf{u} (\mathbf{u} - \mathbf{w}) \cdot \hat{\mathbf{n}} dS + \int_{V_t} \mathbf{f} dV + \oint_{S_t} \mathbf{T} \cdot \hat{\mathbf{n}} dS + \oint_{\text{interface}} (\nabla_S \sigma + 2\mathcal{H}\sigma \hat{\mathbf{n}}) dS, \quad (3.8.12)$$

where the final term represents the surface tension force derived earlier. Applying this equation in the limit of vanishing volume V_t , we obtain

$$\int_1 \mathbf{T} \cdot \hat{\mathbf{n}} dS + \int_2 \mathbf{T} \cdot (-\hat{\mathbf{n}}) dS + \int_{\text{interface}} (\nabla_S \sigma + 2\mathcal{H}\sigma \hat{\mathbf{n}}) dS = 0, \quad (3.8.13)$$

which yields,

$$\mathbf{T} \cdot \hat{\mathbf{n}} \Big|_2 - \mathbf{T} \cdot \hat{\mathbf{n}} \Big|_1 = \nabla_S \sigma + 2\mathcal{H}\sigma \hat{\mathbf{n}} \text{ at the interface.} \quad (3.8.14)$$

The above equation shows that the normal components of traction vector differ due to the curvature of the interface and the tangential components differ due to any surface variation of the surface tension coefficient.

For no surface tension, the dynamic interface condition states that

$$\mathbf{T} \cdot \hat{\mathbf{n}} \Big|_2 = \mathbf{T} \cdot \hat{\mathbf{n}} \Big|_1 \text{ at the interface.} \quad (3.8.15)$$

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