

Chapter 4

A few incompressible laminar flow solutions

In the previous chapter, we derived all the governing equations required for studying classical fluid dynamics. In this course, we are only going to look at incompressible flows, governed by the system of equations,

$$\nabla \cdot \mathbf{u} = 0, \quad (4.0.1)$$

$$\underbrace{\frac{\partial \mathbf{u}}{\partial t}}_{\text{unsteady term}} + \underbrace{\mathbf{u} \cdot \nabla \mathbf{u}}_{\text{convective term}} = - \underbrace{\frac{\nabla p}{\rho}}_{\text{pressure term}} + \underbrace{\nu \nabla^2 \mathbf{u}}_{\text{viscous term}} + \frac{\mathbf{f}}{\rho} \quad (4.0.2)$$

Usual designation of each term is mentioned above. An interesting property of the above system of equations is that the flow is independent of the absolute value of pressure, only the gradient of pressure matters. Additionally, the continuity equation can be seen as a constrained on the velocity. In this chapter, we will look at a few analytical solutions of the above system of equations. Given that the governing equations are nonlinear, there is no unique solution of the equations. A multitude of solutions is possible to a given situation. One of the most easily distinguished classification is between the *turbulent* solutions and *laminar* solutions. Turbulent solutions or flows are characterized by velocity and pressure field that obey the given boundary conditions and initial conditions but exhibit fluctuations in the values away from the boundaries. These fluctuations are so sensitive to uncontrollable factors in an experiment, that it is nearly impossible to reproduce them in detail. Hence, a statistical view point is adopted to tackle these flows. Turbulent flows are inherently unsteady. A detailed introductory discussion is presented in the last chapter.

On the other hand, lamminar solutions or flows can obey the same boundary conditions and exhibit no random fluctuations. They present themselves (under certain conditions) as steady organized flow patterns. In this chapter, we will look at some of these solutions.

4.1 Common assumptions

Before we look at the analytical methods for solving the governing equations Eq.(4.0.2), let us outline a few assumptions.

1. Steady flow: $\frac{\partial}{\partial t} = 0$ for any quantity.
2. Unidirectional flow: $\mathbf{u}(\mathbf{x}, t) = u(x_1, t)\hat{\mathbf{e}}_1$ assuming x_1 is the direction of the flow. The flow is termed as homogeneous in x_2 and x_3 directions.
3. Parallel flow: when $\mathbf{u}(\mathbf{x}, t)$ is always parallel to some external geometry.
4. Planar flow or 2D flow: $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(x_1, x_2, t)$ with flow happening in x_1, x_2 plane. In cylindrical coordinates: $\mathbf{u}(r, \theta, t)$.
5. Axisymmetric flow: $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(r, z, t)$ in cylindrical coordinates. In spherical coordinates: $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(r, \theta, t)$ where θ varies from 0 to π .
6. Periodic flow: $\mathbf{u}(\mathbf{x}, t) = \text{Re}(\hat{\mathbf{u}}(\mathbf{x})e^{i\omega t})$ where ω is the angular frequency of oscillations, $2\pi/\omega$ is the time period. In general, solution of complex $\hat{\mathbf{u}}(\mathbf{x})$ can be determined.
7. Fully developed flow: If flow is fully developed in x_1 , then $\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(x_2, x_3, t)$.

4.2 Fully developed steady flows

The first fully developed flow that we consider is a steady flow driven by pressure in x_1 direction in a channel with a moving wall at $x_2 = h$ (see figure 4.1a). The flow is assumed as homogeneous in x_3 for now (2D/planar flow).

Since the flow is planar and fully developed,

$$\nabla \cdot \mathbf{u} = 0 \implies \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0 \implies u_2 = \text{const.} \quad (4.2.1)$$

Since on both the sides, the flow is bounded by walls, $u_2 = 0$ at $x_2 = \pm h$. Hence,

$u_2 = 0$ and the flow is unidirectional and parallel.

Further, substituting $u_2 = 0$ in x_2 momentum equation, we obtain,

$$\frac{\partial p}{\partial x_2} = 0. \quad (4.2.2)$$

Hence, pressure can only be a function of x_1 . With the simplifications so far, we substitute in the x_1 momentum equation,

$$\cancel{\frac{\partial u_1}{\partial t}} + u_1 \cancel{\frac{\partial u_1}{\partial x_1}} + u_2 \cancel{\frac{\partial u_1}{\partial x_2}} + u_3 \cancel{\frac{\partial u_1}{\partial x_3}} = -\frac{1}{\rho} \frac{dp}{dx_1} + \nu \left(\cancel{\frac{\partial^2 u_1}{\partial x_1^2}} + \frac{\partial^2 u_1}{\partial x_2^2} + \cancel{\frac{\partial^2 u_1}{\partial x_3^2}} \right), \quad (4.2.3)$$

which yields,

$$\frac{dp}{dx_1} = \mu \frac{d^2 u_1}{dx_2^2}. \quad (4.2.4)$$

The LHS is purely a function of x_1 and RHS is purely a function of x_2 . The only way they can be equal is if both are equal to a constant,

$$\frac{dp}{dx_1} = \mu \frac{d^2 u_1}{dx_2^2} = \text{const.} \quad (4.2.5)$$

The only way a laminar fully developed flow can be maintained is through a linear pressure distribution in a channel. Also, for constant pressure gradient,

$$u_1 = \frac{x_2^2}{2\mu} \frac{dp}{dx_1} + C_1 x_2 + C_2, \quad (4.2.6)$$

with the boundary conditions

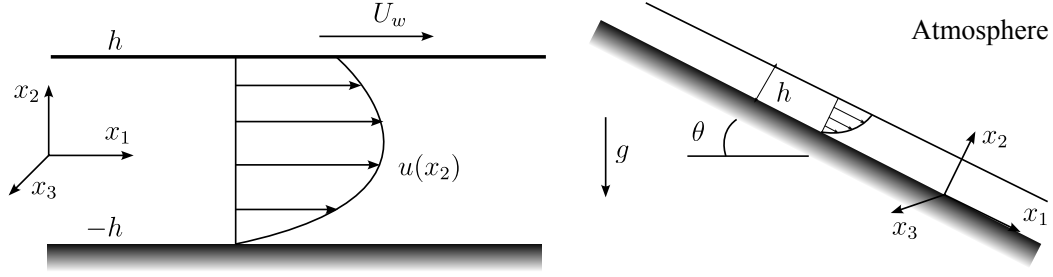
$$u_1(x_2 = h) = U_w, \quad u_1(x_2 = -h) = 0 \implies C_1 = \frac{U_w}{2h}, \quad C_2 = \frac{1}{2} \left(U_w - \frac{h^2}{\mu} \frac{dp}{dx_1} \right). \quad (4.2.7)$$

The final expression for $u_1(x_2)$ is

$$u_1(x_2) = \underbrace{\frac{U_w}{2} \left(1 + \frac{x_2}{h} \right)}_{\text{Couette}} - \underbrace{\frac{1}{2\mu} \frac{dp}{dx_1} (h^2 - x_2^2)}_{\text{Poiseuille}}. \quad (4.2.8)$$

Couette flow is flow induced by only moving wall and Poiseuille flow is flow caused by only pressure gradient. The two solutions linearly superimpose because the governing equation Eq. (4.2.5) for fully developed unidirectional flows is a linear PDE.

As another example of fully developed flows, we consider a falling film down an



(a) Pressure driven unidirectional flow with moving wall. (b) Pressure driven unidirectional flow with moving wall.

inclined plane of angle θ . In this case, the only two changes in Eq. (4.2.5) are due to the gravitational body force term such that,

$$\frac{\partial p}{\partial x_1} = \mu \frac{d^2 u_1}{dx_2^2} + \rho g \sin \theta. \quad (4.2.9)$$

Pressure has hydrostatic variation in x_2 direction. Even though $u_2 = 0$ through same arguments as earlier, presence of gravity results in a variation of pressure in x_2 direction such that,

$$\frac{\partial p}{\partial x_2} = -\rho g \cos \theta \implies p = -\rho g \cos \theta x_2 + f(x_1) + C. \quad (4.2.10)$$

Now, there is no way to move forward from here without considering the interface between the falling fluid and the atmosphere. Let the falling film be of a liquid (such as water) and atmosphere be air. Interface conditions derived in previous chapter tell us that

$$\mathbf{T} \cdot \hat{\mathbf{n}} \Big|_{\text{liquid}} = \mathbf{T} \cdot \hat{\mathbf{n}} \Big|_{\text{air}} \quad \text{at } y = h. \quad (4.2.11)$$

Note that we have ignored any surface tension. The normal at the interface in this case is $\hat{\mathbf{e}}_2$. Hence, we get the following equations,

$$T_{i2} \Big|_{\text{liquid}} = T_{i2} \Big|_{\text{air}}, \quad (4.2.12)$$

which yields for the planar case

$$-p + 2\mu \frac{\partial u_2}{\partial x_2} \Big|_{liquid} = -p + 2\mu \frac{\partial u_2}{\partial x_2} \Big|_{air}, \quad (4.2.13)$$

$$\mu \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \Big|_{liquid} = \mu \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \Big|_{air} \quad (4.2.14)$$

at $x_2 = h$. Since the falling film is unidirectional, the kinematic interface condition yields that air flow can also be only in x_1 direction. We get

$$p = p_{atm} \quad (4.2.15)$$

at $x_2 = h$. This holds for all x_1 values, implying that $\frac{\partial p}{\partial x_1} = 0$ at $x_2 = h$. Hence $f(x_1)$ in Eq. (4.2.10) is constant and pressure can be written as,

$$p = p_{atm} - \rho g \cos \theta x_2. \quad (4.2.16)$$

Furthermore, we note that

$$\mu \frac{\partial u_1}{\partial x_2} \Big|_{liquid} = \mu \frac{\partial u_1}{\partial x_2} \Big|_{air} \quad (4.2.17)$$

at $x_2 = h$. Assuming $\mu_{liquid} \gg \mu_{air}$, we obtain that the interface has negligible shear

$$\frac{\partial u_1}{\partial x_2} \Big|_{liquid} = \frac{\mu_{air}}{\mu_{liquid}} \frac{\partial u_1}{\partial x_2} \Big|_{air} \approx 0. \quad (4.2.18)$$

This yields the second boundary condition to solve for u_1 , with the first boundary condition being of no slip at $x_2 = 0$. Solving Eq. (4.2.9), we obtain,

$$u_1 = \frac{\rho g \sin \theta h^2}{\mu} \left(\frac{x_2}{h} - \frac{x_2^2}{2h^2} \right). \quad (4.2.19)$$

To determine the thickness of the falling film, flow rate or velocity at some location x_2 must be known or measured.

So far, we have been carrying over the less-intuitive index notation. Part of the reason is to maintain continuity with earlier discussions. However, from this point onwards, we will interchangeably use index notation (wherever simplification of calculus operations is required) and the usual notation for Cartesian coordinates i.e. x_1, x_2, x_3 will be

replaced by x, y, z as usual. The corresponding velocity components will be depicted by u, v, w respectively. For cylindrical or spherical coordinates, we maintain the standard notations of r, θ, z and r, θ, ϕ for coordinates and u_r, u_θ, u_z and u_r, u_θ, u_ϕ , respectively.

4.3 Impulsively accelerating plate

Flow caused by an impulsively moving plate is also known as *Rayleigh's problem*. The plate is assumed infinite in x direction, hence the flow can also be solved assuming uniformity in x direction. Notice that this assumption is stricter than a fully developed flow, in which only the velocity is uniform in x . Here, we assume that all flow quantities $\mathbf{u}$ and p are constant in x .

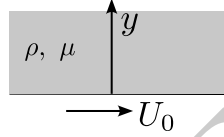


Figure 4.2: Flow induced by a plate which starts moving with constant velocity at $t = 0$.

From continuity equation, we obtain that $v = 0$ upon using the hard wall condition at $y = 0$. This yields $\frac{\partial p}{\partial y} = 0$ (in the absence of gravity).

Substituting the assumptions made so far in x momentum equation, we obtain,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}. \quad (4.3.1)$$

The above equation applies to the domain $y > 0$. It is a second order PDE in space and first order in time. Hence, we need two boundary conditions in space and one initial condition. We assume that the fluid was initially at rest. Furthermore, we assume that the domain extends to infinity in y and the fluid at $y \rightarrow \infty$ remains at rest since the effect of the plate will take very long to reach there. Furthermore, the no slip boundary condition at $y = 0$ holds. This yields the following conditions subject to which Eq. (4.3.1) has to be solved,

$$\underbrace{u(y=0, t) = U_0, \quad u(y \rightarrow \infty, t) = 0}_{\text{Boundary Conditions}}, \quad \underbrace{u(y, t=0) = 0}_{\text{Initial Condition}}. \quad (4.3.2)$$

There are multiple ways in which Eq. (4.3.1) can be solved. Given the domain is *semi-infinite* (unbounded on one side), a well known technique to solve is to use

the concept of the similarity solution. To this end, consider $u(\sqrt{\alpha y}, \alpha t)$ for some $\alpha > 0$. We find that if $u(y, t)$ satisfies the equation, then so does $u(\sqrt{\alpha y}, \alpha t)$. The arguments for both are related such that $\frac{\text{first argument}}{\sqrt{\text{second argument}}}$ remains constant. Hence, we look for the solution by looking for

$$f(\eta) = u(y, t), \text{ where } \eta = \frac{cy}{\sqrt{t}}. \quad (4.3.3)$$

This η is called the similarity variable. The success of this transformation is decided if we are able to convert the PDE in y, t into an ODE in η . To this end, we substitute Eq. (4.3.3) in Eq. (4.3.1) to obtain,

$$f'' + \frac{y}{2c\nu\sqrt{t}}f' = 0. \quad (4.3.4)$$

The solution $u(y, t)$ doesn't depend on c since the expression for both η and $f(\eta)$ change accordingly. The reader is encouraged to retrace below algebra steps for an arbitrary c to confirm this. To make writing simpler, we choose $c = \frac{1}{2\sqrt{\nu}}$ such that,

$$f'' + 2f'\eta = 0, \text{ where } \eta = \frac{y}{2\sqrt{\nu t}}. \quad (4.3.5)$$

The three conditions in Eq. (4.3.2) can be simplified for η as,

$$f(\eta = 0) = U_0, f(\eta \rightarrow \infty) = 0. \quad (4.3.6)$$

The equation can be solved for f' and then for f to yield,

$$f' = Ae^{-\eta^2} \implies f(\eta) - f(0) = A \int_0^\eta e^{-s^2} ds \implies f(\eta) = U_0 + A \int_0^\eta e^{-s^2} ds. \quad (4.3.7)$$

Using the $f(\eta \rightarrow \infty) = 0$ condition, we can determine A as,

$$A \int_0^\infty e^{-s^2} ds = -U_0. \quad (4.3.8)$$

The improper integral $\int_0^\infty e^{-s^2} ds$ has a constant finite value which can be obtained following the steps,

$$I = \int_0^\infty e^{-s^2} ds \implies I^2 = \iint_0^\infty e^{-s^2-l^2} ds dl = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta = \frac{\pi}{4}. \quad (4.3.9)$$

Hence, we obtain for $u(y, t)$,

$$u = U_0 \left(1 - \frac{2}{\sqrt{\pi}} \int_0^{y/2\sqrt{\nu t}} e^{-s^2} ds \right) = U_0 \left(1 - \operatorname{erf} \left(\frac{y}{2\sqrt{\nu t}} \right) \right). \quad (4.3.10)$$

4.3.1 Solution through Laplace transform

Let $U(y, s)$ be the Laplace transform of $u(y, t)$. Then the governing equation reduces to,

$$\frac{sU}{\nu} = \frac{\partial^2 U}{\partial y^2} \implies U = Ae^{\sqrt{\frac{s}{\nu}}y} + Be^{-\sqrt{\frac{s}{\nu}}y}. \quad (4.3.11)$$

Since $u(y \rightarrow \infty, t) = 0$, A has to be 0 in the above equation. We are left with,

$$U = Be^{-\sqrt{\frac{s}{\nu}}y} \text{ with condition } U(y=0, t) = \frac{U_0}{s}, \quad (4.3.12)$$

since Laplace transform of constant c is c/s . This yields the solution

$$U = \frac{U_0}{s} e^{-\sqrt{\frac{s}{\nu}}y}. \quad (4.3.13)$$

The reader is encouraged to see how to evaluate the inverse Laplace transform using contour integration and verify the solution as obtained above. Note that the only pole is at $s = 0$.

4.4 Laminar flow outside an impulsively rotating cylinder

Cylindrical analogue of the Rayleigh's problem is that of the flow outside an impulsively rotating cylinder. We assume that the flow is *axisymmetric*, i.e. $\frac{\partial \mathbf{u}}{\partial \theta} = 0$. Since the cylinder is infinitely long, all flow properties are homogeneous in z .



Figure 4.3: Flow induced by an infinitely long cylinder which starts rotating with constant angular velocity at $t = 0$.

Applying the continuity equation, we obtain,

$$\frac{1}{r} \left(\frac{\partial(ru_r)}{\partial r} \right) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} = 0 \implies ru_r = \text{const.} \quad (4.4.1)$$

We take the cylinder to be made of solid walls. Hence $u_r = 0$ at $r = R$, which yields $u_r = 0$ throughout. Hence, the flow is purely unidirectional with only u_θ component. Additionally, since the flow is homogeneous in z and axisymmetric by assumptions, $u_\theta(r, t)$ is just a function of r and t . Substituting the above in r momentum equation,

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right), \quad (4.4.2)$$

where

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}. \quad (4.4.3)$$

We obtain,

$$\frac{\partial p}{\partial r} = \frac{\rho u_\theta^2}{r} \implies p_\infty - p = \int_r^\infty \frac{\rho u_\theta^2}{r} dr + f(\theta). \quad (4.4.4)$$

Since the pressure at infinity will be constant in θ (fluid will still be at rest at $r \rightarrow \infty$), we conclude that $f(\theta) = \text{const.}$ which may be absorbed in p_∞ . Hence $\frac{\partial p}{\partial \theta} = 0$, which is useful in θ momentum equation,

$$\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \nu \left(\nabla^2 u_\theta + \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} - \frac{u_\theta}{r^2} \right). \quad (4.4.5)$$

Upon simplification, we obtain the equation,

$$\frac{\partial u_\theta}{\partial t} = \nu \left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} \right). \quad (4.4.6)$$

The boundary and initial conditions are,

$$u_\theta(r = R, t) = \Omega R, \quad u_\theta(r \rightarrow \infty, t) = 0, \quad u_\theta(r, t = 0) = 0. \quad (4.4.7)$$

Finding a solution to this equation in terms of self similarity variable $\eta = r/\sqrt{t}$ is possible. However, as it is discovered $u_\theta = t^{-\gamma} g(\eta)$ yields an ODE for any arbitrary γ and the solutions have to be obtained for all γ values which makes the process cumbersome. Fortunately, the separation of variables technique offers

an easier way. Assuming

$$u_\theta = T(t)f(r), \quad (4.4.8)$$

we obtain,

$$\frac{T'}{T} = \frac{\nu}{f} \left(f'' + \frac{f'}{r} - \frac{f}{r^2} \right). \quad (4.4.9)$$

Since LHS is purely a function of t and RHS is purely a function of r , equality can hold when both are equal to a constant. Let the constant be $\lambda\nu$, which yields

$$T = Ae^{\lambda\nu t}; r^2 f'' + r f' - (\lambda r^2 + 1)f = 0. \quad (4.4.10)$$

Note that we have not assumed anything about λ . It may be 0 or even an imaginary number. In these type of equations, the possible values of λ are obtained by noting down for which possible values, the overall solution satisfies the boundary conditions.

Case: $\lambda = 0$

For $\lambda = 0$, the solution is

$$T = \text{const.}; f = B_0 r + \frac{C_0}{r}. \quad (4.4.11)$$

Case: $\lambda \neq 0$

For $\lambda \neq 0$, we have the following choices. Real $\lambda > 0$ would yield a solution exponentially growing in time resulting in unbounded velocity which is not possible. Imaginary λ would result in an oscillatory solution, which is also not possible given the boundary condition is constant in time. Hence, we conclude that only physically possible values of $\lambda < 0$. Let $\lambda = k^2$ where k is purely imaginary, then

$$T = A_k e^{-k^2 \nu t}; f = B_k J_1(kr) + C_k Y_1(kr), \quad (4.4.12)$$

where $J_1(x)$ and $Y_1(x)$ are first order Bessel's functions of first and second kind, respectively.

Full solution

Full solution is a linear combination of the above two solutions. However, in the $\lambda > 0$ case, we note that both $J_1(kr)$ and $Y_1(kr)$ approach 0 as $r \rightarrow \infty$. This means that the linear combination has to be taken over a **continuous range**

of k values ranging from $-\infty$ to ∞ . This is a standard behavior of problems in cylindrical domains where inner radius is finite and non-zero but the domain is unbounded outside this radius. Hence, the complete solution to the equation with prescribed boundary conditions reads,

$$u_\theta = B_0 r + \frac{C_0}{r} + \int_{-\infty}^{\infty} e^{-k^2 \nu t} (B(k) J_1(kr) + C(k) Y_1(kr)) dk, \quad (4.4.13)$$

where $B(k) = A_k B_k$ and $C(k) = A_k C_k$. Satisfying the boundary condition at $r \rightarrow \infty$, we obtain $B_0 = 0$. Satisfying the boundary condition at $r = R$, we get,

$$\Omega R = \frac{C_0}{R} + \int_{-\infty}^{\infty} e^{-k^2 \nu t} (B(k) J_1(kR) + C(k) Y_1(kR)) dk. \quad (4.4.14)$$

Since the LHS is independent of time, we get,

$$C_0 = \Omega R^2, \quad B(k) J_1(kR) + C(k) Y_1(kR) = 0 \implies C(k) = -B(k) \frac{J_1(kR)}{Y_1(kR)}. \quad (4.4.15)$$

Using this, one may write the solution as,

$$u_\theta(r, t) = \frac{\Omega R^2}{r} + \int_{-\infty}^{\infty} A(k) e^{-k^2 \nu t} (J_1(kr) Y_1(kR) - Y_1(kr) J_1(kR)) dk. \quad (4.4.16)$$

In the above solution $A(k)$ function remains to be determined. This can be determined by satisfying the initial condition,

$$\int_{-\infty}^{\infty} A(k) (J_1(kr) Y_1(kR) - Y_1(kr) J_1(kR)) dk = -\frac{\Omega R^2}{r}, \quad (4.4.17)$$

or

$$2 \int_0^{\infty} A(k) (Y_1(kr) J_1(kR) - J_1(kr) Y_1(kR)) dk = \frac{\Omega R^2}{r}. \quad (4.4.18)$$

Here we use a “well-known” result in integration results, the so-called *Weber integral*

$$2 \int_0^{\infty} \frac{Y_1(kr) J_1(kR) - J_1(kr) Y_1(kR)}{J_1^2(kR) + Y_1^2(kR)} \frac{dk}{k} = \frac{\pi R}{r}. \quad (4.4.19)$$

Stuff like this will be given as *useful formulae* in exams. If not given, you can expect full points till usage of this result. Those who are able to push-through ahead in case no formula is given will get bonus points. From Weber's integral,

we obtain,

$$A(k) = \frac{\Omega R}{\pi k} \frac{1}{J_1^2(kR) + Y_1^2(kR)}, \quad (4.4.20)$$

which completes the solution as,

$$u_\theta(r, t) = \frac{\Omega R^2}{r} + \frac{2\Omega R}{\pi} \int_0^\infty e^{-k^2 \nu t} \frac{J_1(kr)Y_1(kR) - Y_1(kr)J_1(kR)}{J_1^2(kR) + Y_1^2(kR)} \frac{dk}{k}. \quad (4.4.21)$$

There is another counter-part to the Rayleigh's problem in cylindrical coordinates, flow of a fluid inside a cylinder which starts to rotate impulsively. In this problem, the domain is bounded between $0 < r < R$ and the solution is much simpler since the eigenvalues obtained are discrete.

4.4.1 Solution through Laplace transform

Let $U(r, s)$ be the Laplace transform of $u_\theta(r, t)$. The governing equation reduces to,

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} - \frac{U}{r^2} - \frac{s}{\nu} U = 0. \quad (4.4.22)$$

Writing $s/\nu = p^2$, the general solution of the above equation is,

$$U = AI_1(pr) + BK_1(pr), \quad (4.4.23)$$

where $I_1(z)$ and $K_1(z)$ are the modified Bessel functions of first and second kind, respectively. Since $I_1(z)$ is unbounded as $|z| \rightarrow \infty$, the general solution for flow outside the cylinder is,

$$U = BK_1(pr). \quad (4.4.24)$$

Since $u_\theta(R, t) = \Omega R$, the Laplace transformed boundary condition is,

$$U(R) = \frac{\Omega R}{s}, \quad (4.4.25)$$

which yields,

$$U = \frac{\Omega R}{s} \frac{K_1(pr)}{K_1(pR)}. \quad (4.4.26)$$

To invert this, one can use the inversion formula

$$u_\theta = \frac{\Omega R}{2\pi i} \oint_C \frac{1}{s} \frac{K_1(pr)}{K_1(pR)} e^{st} ds, \quad (4.4.27)$$

where C is a suitable contour (see [Goldstein \[1932\]](#)).

4.5 Steady state laminar flow through pipes

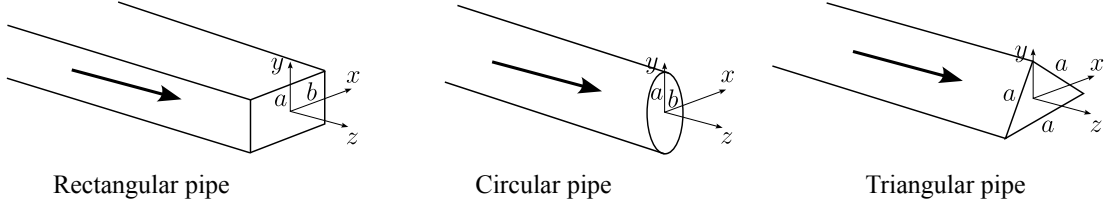


Figure 4.4: Cross-section of different cross-section pipes.

Assume that flow through a pipe (of any cross-section) is steady and unidirectional. Let the z axis aligned along the flow direction as shown in figure 4.4. Since the flow is unidirectional (only w is possible), $u = v = 0$. Applying the continuity equation, we obtain,

$$\frac{\partial w}{\partial z} = 0. \quad (4.5.1)$$

The flow is fully developed. From x and y momentum equations, we obtain,

$$\frac{\partial p}{\partial x} = 0, \quad \frac{\partial p}{\partial y} = 0. \quad (4.5.2)$$

Substituting in z momentum equation, we obtain,

$$\frac{\partial p}{\partial z} = \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \implies \frac{dp}{dz} = \mu \nabla^2 w, \quad (4.5.3)$$

where ∇^2 is 2D Laplacian. The boundary conditions are that $w = 0$ on the walls. Based on the geometry, these boundary conditions are applied on a different surface, resulting in different solutions. Equation (4.5.3) is a Poisson equation. Standard technique to solve a Poisson equation in a bounded domain is to convert the dependent variable such that the equation becomes Laplacian. Furthermore, note that $dp/dz = \text{const.}$. Considering,

$$w = \psi - \frac{dp}{dz} \frac{1}{4\mu} (x^2 + y^2), \quad (4.5.4)$$

we obtain that ψ satisfies $\nabla^2 \psi = 0$ with boundary conditions,

$$\psi = \frac{dp}{dz} \frac{1}{4\mu} (x^2 + y^2) \text{ on walls of the cross-section.} \quad (4.5.5)$$

Laplacian of $C_1 x^2 + C_2 y^2$ with addition of any degree two multiplication of x and y is $C_1 + C_2 = \text{const.}$. Hence, a more general way of obtaining a ψ which

converts the governing equation into a Laplacian equation is to add/subtract to w , $C_1x^2 + C_2y^2$ where $C_1 + C_2 = \frac{1}{2\mu} \frac{dp}{dz}$.

4.5.1 Using complex variables

If a complex function $F(z) = \phi + i\psi$, where $Z = x + iy$ is twice differentiable, then, from Cauchy Riemann equations, we have,

$$\frac{dF}{dZ} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = -i\frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial y} \implies \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \text{ and } \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}. \quad (4.5.6)$$

Differentiating the above, we note that both ϕ and ψ satisfy the Laplacian equation. Hence, real and imaginary parts of any twice differentiable complex function satisfy the Laplacian equation. To satisfy boundary conditions, we must satisfy $w = 0$ on the boundary. Let the boundary be defined as the curve $f(x, y) = 0$. If we can establish that $f(x, y)$ has constant Laplacian, then we can construct solutions from boundaries.

Elliptical pipe

For an ellipse, the boundary is

$$1 - \frac{x^2}{b^2} - \frac{y^2}{a^2} = 0. \quad (4.5.7)$$

Naturally, in this case, the Laplacian is constant for the boundary function. Hence, we can directly set

$$w = C \left(1 - \frac{x^2}{b^2} - \frac{y^2}{a^2} \right). \quad (4.5.8)$$

Substituting in the governing equation, we find

$$C = -\frac{1}{2\mu} \frac{dp}{dz} \frac{a^2b^2}{a^2 + b^2}. \quad (4.5.9)$$

Triangular pipe

Consider a pipe with equilateral triangle as its cross-section. Let the side length be a and its base sitting at $y = 0$. In such a case, the cross-section boundary is defined by,

$$f = y \left(x + \frac{y}{\sqrt{3}} - \frac{a}{2} \right) \left(x - \frac{y}{\sqrt{3}} + \frac{a}{2} \right) = 0. \quad (4.5.10)$$

We note that this function is

$$f = \frac{\text{Im}(Z^3)}{3} + \frac{ay^2}{\sqrt{3}} - \frac{a^2y}{4}. \quad (4.5.11)$$

Since Z^3 is a twice differentiable function, we get that $\nabla^2 f = \text{const.}$. Hence,

$$w = Cf = Cy \left(x^2 - \left(\frac{y}{\sqrt{3}} - \frac{a}{2} \right)^2 \right) \quad (4.5.12)$$

is a solution which satisfied the boundary condition. C is related to pressure gradient as,

$$C = \frac{\sqrt{3}}{2a} \frac{1}{\mu} \frac{dp}{dz}. \quad (4.5.13)$$

Constructing shapes from analytical functions

See instructional discussion in [Lekner \[2007\]](#). We can consider some analytic functions of Z which are twice differentiable. Isolating the real or imaginary part of these functions (which satisfy the Laplacian) and adding a function whose Laplacian is constant (such as $C_1x^2 + C_2y^2 + C_3xy + C_4x + \dots$), we can construct solutions for flow through pipes with arbitrary cross-sections.

4.5.2 Using separation of variables

For Cartesian geometry such as a rectangular cross-section, doing separation of variables is straightforward. Let us consider a rectangular cross-section pipe with x axis length $2b$ and y axis length $2a$. The equation to solve is,

$$\nabla^2 w = \frac{1}{\mu} \frac{dp}{dz}. \quad (4.5.14)$$

As discussed above, we can convert this to a homogeneous equation with inhomogeneous boundary conditions. Since we anticipate application of Fourier integrals, we chose one of C_1 or C_2 to be 0, for easier boundary condition implementation. Let

$$w = \frac{1}{2\mu} \frac{dp}{dz} (y^2 - a^2) + \psi, \quad (4.5.15)$$

such that

$$\nabla^2 \psi = 0, \quad (4.5.16)$$

and

$$\psi = 0 \text{ at } y = \pm a \text{ and } \psi = -\frac{1}{2\mu} \frac{dp}{dz} (a^2 - y^2) \text{ at } x = \pm b. \quad (4.5.17)$$

Using variable separation, we substitute $\psi = X(x)Y(y)$ to obtain,

$$\frac{Y''}{Y} + \frac{X''}{X} = 0 \implies \frac{Y''}{Y} = -\lambda^2 \text{ and } \frac{X''}{X} = \lambda^2, \quad (4.5.18)$$

for an arbitrary λ . This yields general solutions,

$$Y = Ae^{i\lambda y} + Be^{-i\lambda y}, \quad X = Ce^{\lambda x} + De^{-\lambda x}. \quad (4.5.19)$$

Substituting the boundary conditions,

$$\psi = 0 \text{ at } y = \pm a \implies Y(\pm a) = 0. \quad (4.5.20)$$

Since Y has to be symmetric, $A = B$. Also,

$$2A \cos(\lambda a) = 0 \implies \lambda = (2n+1) \frac{\pi}{2a}. \quad (4.5.21)$$

Hence, only a discrete set of λ_n satisfy the boundary conditions in y . Hence, a complete solution is obtained by a linear combination (since the governing equation is linear) of solutions constructed using all such possible values. We write,

$$\psi = \sum_n \cos(\lambda_n y) (\alpha_n e^{\lambda_n x} + \beta_n e^{-\lambda_n x}). \quad (4.5.22)$$

Since the flow is symmetric $\alpha_n = \beta_n$. To obtain their values, we write the boundary conditions at $x = \pm b$,

$$-\frac{1}{2\mu} \frac{dp}{dz} (a^2 - y^2) = \sum_n 2\alpha_n \cos(\lambda_n y) \cosh(\lambda_n b). \quad (4.5.23)$$

Using the orthogonality of $\cos(\lambda_n y)$,

$$\int_{-a}^a \cos(\lambda_n y) \cos(\lambda_m y) dy = a \delta_{mn}, \quad (4.5.24)$$

we multiply Eq. (4.5.23) with $\cos(\lambda_m y)$ and integrate over the y domain to get,

$$-\frac{1}{2\mu} \frac{dp}{dz} \int_{-a}^a (a^2 - y^2) \cos(\lambda_m y) dy = 2 \sum_n \alpha_n \cosh(\lambda_n b) \int_{-a}^a \cos(\lambda_n y) \cos(\lambda_m y) dy, \quad (4.5.25)$$

which yields,

$$2a\alpha_m \cosh(\lambda_m b) = -\frac{1}{2\mu} \frac{dp}{dz} \int_{-a}^a (a^2 - y^2) \cos(\lambda_m y) dy = -\frac{1}{\mu} \frac{dp}{dz} \frac{2(-1)^m}{\lambda_m^3}. \quad (4.5.26)$$

This yields the solution for w as,

$$w = \frac{1}{2\mu} \frac{dp}{dz} (y^2 - a^2) - \frac{2}{\mu a} \frac{dp}{dz} \sum_{n=0}^{\infty} \frac{(-1)^n}{\lambda_n^3} \cos(\lambda_n y) \frac{\cosh(\lambda_n y)}{\cosh(\lambda_n b)}. \quad (4.5.27)$$

4.6 Laminar flow over an oscillating plate

As another time dependent problem, we consider laminar flow over an oscillating plate. This problem was considered by Stokes and is useful in modeling a variety of flows, including propagation of acoustic waves near solid boundaries (in compressible flows). We will assume homogeneity or uniformity in x and that the

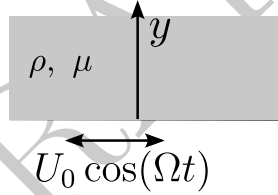


Figure 4.5: Stokes's problem of laminar flow over an oscillating plate

fluid extends to $y \rightarrow \infty$. Due to uniformity in x , we again get from continuity equation that $v = 0$ upon using the hard wall condition. This yields $\frac{\partial p}{\partial y} = 0$ in the absence of gravity. Substituting these assumption, we obtain

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \quad (4.6.1)$$

subject to the boundary and initial conditions,

$$u(y = 0, t) = U_0 \cos(\Omega t), u(y \rightarrow \infty, t) = 0, u(y, t = 0) = 0. \quad (4.6.2)$$

The flow is initially at rest (quiescent). The plate starts to oscillate and there is a transient flow which eventually gives way to steady state flow. We will calculate both below.

4.6.1 Steady state

At steady state, the flow is expected to oscillate at frequency Ω with some other features. Hence, to solve for steady state, it is prudent to use complex variables, just like we find steady state oscillations of a forced spring-mass-damper system. We will consider

$$u = U(y)e^{i\Omega t}, \quad (4.6.3)$$

with the obvious understanding that the velocity is actually the real part of the above and U is complex amplitude which varies in y . The boundary conditions on $U(y)$ will be,

$$U(y \rightarrow \infty) = 0, U(y = 0) = U_0. \quad (4.6.4)$$

Substituting in the governing equation, we obtain,

$$\frac{i\Omega}{\nu}U = \frac{d^2U}{dy^2}, \quad (4.6.5)$$

which yields the solution,

$$U(y) = Ae^{\eta y} + Be^{-\eta y}, \quad (4.6.6)$$

where $\eta = \sqrt{\frac{i\Omega}{\nu}} = (1+i)\sqrt{\frac{\Omega}{2\nu}}$. Substituting for the boundary conditions, we obtain,

$$A = 0, B = U_0. \quad (4.6.7)$$

This yields the solution,

$$U(y) = U_0e^{-\eta y}, \quad (4.6.8)$$

and finally,

$$u(y, t) = U_0e^{-\eta y + i\Omega t} = U_0e^{-\frac{y}{\delta}}e^{i(\Omega t - \frac{y}{\delta})}, \quad (4.6.9)$$

where $\delta = \sqrt{\frac{2\nu}{\Omega}}$. The solution is obviously the real part of Eq. (4.6.9). An interesting thing about steady state oscillatory solution is that, we did not use the initial condition anywhere. Notice how the solution is completely unique without using the initial condition. This is because, from the beginning we assumed a solution of the form $Ue^{i\Omega t}$ which is a purely oscillatory solution with constant frequency and amplitude (at a given y).

4.6.2 Transient

To determine the transient solution, we consider

$$u = u_S + u_T = U_0 e^{-\frac{y}{\delta}} \operatorname{Re} \left(e^{i(\Omega t - \frac{y}{\delta})} \right) + u_T, \quad (4.6.10)$$

here u_T satisfies the condition such that $u(y, t = 0) = 0$. This yields,

$$u_T(y, t = 0) = -U_0 e^{-\frac{y}{\delta}} \operatorname{Re} \left(e^{i(\Omega t - \frac{y}{\delta})} \right) = -U_0 e^{-\frac{y}{\delta}} \cos \left(\Omega t - \frac{y}{\delta} \right), \quad (4.6.11)$$

combined with the boundary conditions $u_T(y \rightarrow \infty, t) = 0$ and $u_T(y = 0, T) = 0$, since u_S already satisfies the no-slip boundary condition at the oscillating plate. The equation to solve is

$$\frac{\partial u_T}{\partial t} = \nu \frac{\partial^2 u_T}{\partial y^2} \text{ in } y > 0. \quad (4.6.12)$$

To solve the transient solution with the prescribed initial condition, we use the *Fourier sine transform*, which is Fourier transform for odd functions. We consider that the domain is extended for $y < 0$ such that the velocity satisfies $u_T(-y) = -u_T(y)$, an odd function. This way, $u_T(y = 0)$ is always maintained at 0, thus ensuring that the $y = 0$ boundary condition is always satisfied. The Fourier sine transform and inverse for a function $f(y)$ are defined as,

$$\hat{f}(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(y) \sin(ky) dy, \quad f(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}(k) \sin(ky) dk. \quad (4.6.13)$$

From the above definitions, we see that the sine transform of $\frac{d^2 f}{dy^2}$ is given by $-k^2 \hat{f}$. Using this, we apply sine transform to Eq. (4.6.12) to obtain,

$$\frac{\partial \hat{u}}{\partial t} = -\nu k^2 \hat{u} \quad (4.6.14)$$

which yields the solution $\hat{u} = \hat{u}(t = 0) e^{-\nu k^2 t}$. The initial condition $\hat{u}(t = 0)$ is obtained as,

$$\hat{u}(t = 0) = -U_0 \sqrt{\frac{2}{\pi}} \int_0^\infty \cos \left(\Omega t - \frac{p}{\delta} \right) \sin(kp) dp, \quad (4.6.15)$$

where we have replaced the integration variable with p to avoid confusion. Substituting the above, we find the solution,

$$u_T(y, t) = -\frac{2U_0}{\pi} \int_0^\infty \left(\int_0^\infty \cos\left(\Omega t - \frac{p}{\delta}\right) \sin(kp) dp \right) e^{-\nu k^2 t} \sin(ky) dk. \quad (4.6.16)$$

Another interesting part of the above solution is $\delta = \sqrt{2\nu/\Omega}$. We see this “length” appearing automatically in the solution. If we look at the governing equation, we see that the flow is dependent on the frequency Ω and the kinematic viscosity of the fluid ν . Using these two parameters, we note that a *characteristic length scale* of the problem is $\sim \sqrt{\Omega/\nu}$. The concept of characteristic scales (length, time, velocity, etc) yields useful information about the flow. For instance, if in a situation $\sqrt{\Omega/\nu} \approx 1$ cm, then using screw-guage for measurement in such a flow would be useless! So would be using a scale with least count of 1 m! Characteristic scales tell us what are the physical orders of magnitude of the particular quantity we are dealing with. For instance, characteristic scale of interatomic distance is 10^{-10} m, characteristic scale of velocity of a road car is 10 m/s, characteristic length scale of atmospheric flow may be 1 km, depending on the phenomenon of interest, characteristic temperature scale of human body is 30° C.

In the next chapter, we will use these characteristic scales to derive dimensionless numbers and dimensionless form of governing equations.