

Chapter 5

Dimensional analysis

In this chapter, we will look at the non-dimensionalization of the governing equations, emerging dimensionless numbers and their physical significance, and eventually how to separate regimes of governing equations based on very low or very high values of these dimensionless numbers.

5.1 Characteristic scales

The concept of characteristic scales, or reference scales informs about the typical values (with units) of various flow parameters which are expected. For instance, velocity scale of a flow past a commercial flight will be $0.3 \times$ the speed of sound or 100 m/s. Velocity scale of flow of gases in a star-explosion might be some fraction of the speed of light! Similarly, velocity scale of motion of a glacier might be $1 \mu\text{m/s}$. Similarly, you can think about the length scales and time scales of various flows. In this course, we will use the following notation. To denote variables of characteristic scales, we will use the square brackets as $[\cdot]$. For instance, if velocity scale in all directions in a flow is U_0 , we will write it as,

$$[\mathbf{u}] = U_0. \quad (5.1.1)$$

Similarly, $[\mathbf{x}] = d$ denotes that the length scale in all directions is d . **There is no unique choice of characteristic scales. But there are few convenient choices.** For instance, to measure distance travelled by an airplane, a meter scale can be technically used, we will just be counting till 10^6 ! Hence, a convenient choice might be kilometers or megameters. Similarly, to measure the time taken for sound to reach another room in a house, one could use a stop-watch, but only

if one possesses an insanely fast reaction time! In fluid mechanics, flows such as turbulent flows have multiple length scales, time scales, and velocity scales associated with them. Using one particular scale, defines the regime in which analysis/study is being done. This will become clearer as we move further.

Besides characteristic scales, we will also use the $\mathcal{O}()$ notation in this course to denote *orders of magnitude*. This notation can be used for both dimensionless or dimensional variables. For instance, time scale of a flow can be written as $t \sim \mathcal{O}(1)$ s.

5.2 Dimensionless variables, equations, and numbers

It is very informative to look at the governing equation in a dimensionless form. To this end, we define dimensionless variables. Dividing the flow variables ($\mathbf{u}$ and p for incompressible flow) and the independent variables $\mathbf{x}$ and t with respective characteristic scales or reference scales, we obtain the dimensionless values. While deriving, we will often use $()^*$ notation to denote a dimensionless variable. For instance,

$$\mathbf{u}^* = \mathbf{u}/U_0, \quad (5.2.1)$$

defines a dimensionless velocity. Similarly, $t^* = t/\tau$ defines a dimensionless time. Using these, we can non-dimensionalize the governing equations as done below.

5.2.1 Dimensionless Navier-Stokes equation

Let us consider a flow with characteristic length scale ℓ in all directions, characteristic velocity scale U_0 in all directions, of a fluid with constant density ρ and viscosity μ . Let the pressure be defined using some scale p_0 . Defining the dimensionless velocity, pressure, space, and time, as,

$$\mathbf{x}^* = \frac{\mathbf{x}}{\ell}, \quad \mathbf{u}^* = \frac{\mathbf{u}}{U_0}, \quad t^* = \frac{tU_0}{\ell}, \quad (5.2.2)$$

we can write the following form of dimensionless governing equations for incompressible flow,

$$\frac{U_0}{\ell} \nabla^* \cdot \mathbf{u}^* = 0 \implies \nabla^* \cdot \mathbf{u}^* = 0, \quad (5.2.3)$$

$$\frac{U_0^2}{\ell} \left(\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* \right) = -\frac{p_0}{\rho \ell} \nabla^* p^* + \frac{\nu U_0}{\ell^2} \nabla^{*2} \mathbf{u}^*, \quad (5.2.4)$$

where $\nabla^* = \ell \nabla$. Simplifying the above equation, we obtain the following dimensionless momentum equation,

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* = -\frac{p_0}{\rho U_0^2} \nabla^* p^* + \frac{\mu}{\rho U_0 \ell} \nabla^{*2} \mathbf{u}^*. \quad (5.2.5)$$

We notice the following two dimensionless numbers in the above equation,

$$\text{Eu} = \frac{p_0}{\rho U_0^2}, \quad \text{Re} = \frac{\rho U_0 \ell}{\mu}, \quad (5.2.6)$$

where Eu denotes the Euler number and Re denotes the Reynolds number.

5.2.2 Dimensionless numbers

The dimensionless numbers introduced above (Eu and Re) carry special meanings. In terms of characteristic scales, they define the ratio of competing effects in flows. For instance, pressure in a flow may vary due to high viscous stresses or high speed. In both the cases, the pressure scale p_0 can be formulated using the relevant scales.

For instance, if a relevant pressure scale in the problem is a viscous pressure scale, then

$$p_0 = \frac{\mu U_0}{\ell} \implies \text{Eu} = \frac{1}{\text{Re}}. \quad (5.2.7)$$

Conversely, if the relevant scale is inertial type (due to the movement of the fluid particles),

$$p_0 = \rho U_0^2 \implies \text{Eu} = 1. \quad (5.2.8)$$

Determining which is relevant depends on the problem. For instance a steady flow driven by pressure forces in which the flow speed is very small, we would expect that the pressure forces would tend to balance viscous stresses, given that the inertial stresses ($\rho \mathbf{u} \cdot \nabla \mathbf{u}$) will be very small. The terminology of *small* in the previous statement is incomplete. Small-ness or large-ness of any quantity must be specified *in comparison to some other quantity*. We will formalize this more

when we discuss flow regime classification based on dimensionless numbers.

The Reynolds number carries very special meaning in fluid flows. By definition, it may be looked at as a ratio of characteristic *inertial forces to viscous forces*,

$$\text{Re} = \frac{\rho U_0 \ell}{\mu} = \frac{\rho U_0^2 \ell^2}{\mu U_0 \ell} = \frac{\text{characteristic inertial force}}{\text{characteristic viscous force}}, \quad (5.2.9)$$

or as a ratio of characteristic time scales as,

$$\text{Re} = \frac{\rho U_0 \ell}{\mu} = \frac{\ell^2 / \nu}{\ell / U_0} = \frac{\text{viscous time scale}}{\text{inertial time scale}}. \quad (5.2.10)$$

It may also be cast in terms of the ratio of velocity scales or length scales. A small Reynolds number ($\text{Re} \ll 1$) means the viscous forces dominate the inertial forces or the inertial time scales are very long (slow) compared to the viscous time scales. Conversely, a large Reynolds number ($\text{Re} \gg 1$) denotes that the inertial time scales are very short (fast) compared to the viscous time scales (and similarly in terms of forces).

Choosing convenient characteristic scales now implies choosing the characteristic scales, which make the study of a particular problem simple. If the scales are not chosen correctly, utmost care must be practiced while dismissing some values as *very large* or *very small*.

For instance, in our derivation of dimensionless momentum Eq. (5.2.4), we silently used the characteristic scales of quantities, for characteristic scales of changes in quantities. This is not always the case, as demonstrated below in the derivation of the Boussinesq approximation.

5.2.3 The Boussinesq approximation - revisited

In Chapter 3, we introduced the Boussinesq system of equations (see momentum Eq. (5.2.28) and surrounding discussion) and promised a thorough derivation. Consider flow due to bouyancy effects generated under gravity and density gradients.

Let the characteristic scales for *change in density* be $\Delta\rho$, for density ρ_0 , for pressure p_0 , for length ℓ . Since the velocity of the flow is due to density changes, we can use the standard dimensional analysis to develop a characteristic velocity

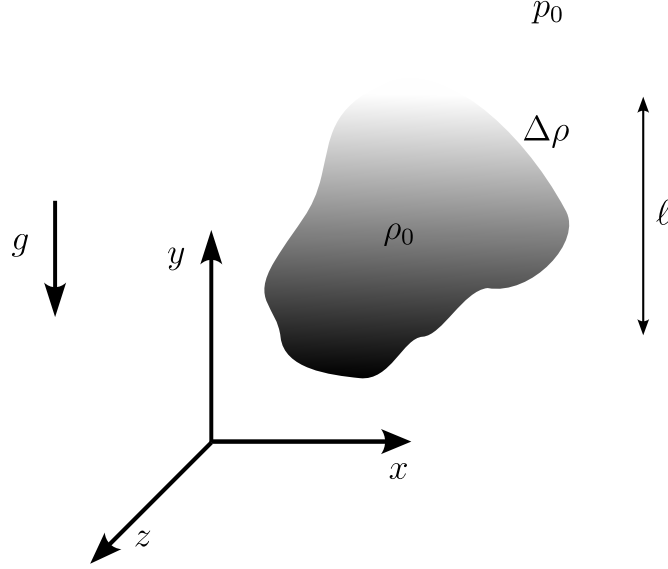


Figure 5.1: Flow caused by density changes in a gravitational field (natural convection) with density changes caused by temperature changes only.

scale in the flow,

$$U = f(\Delta\rho, g, \ell, \rho_0) \implies U = \sqrt{\frac{\Delta\rho}{\rho_0}} g\ell. \quad (5.2.11)$$

Additionally, the pressure scale p_0 can be related to the density and length scale as,

$$p_0 = f(\rho_0, \ell, g) = \rho_0 \ell g. \quad (5.2.12)$$

Let us non-dimensionalize the continuity equation. We obtain,

$$\frac{\rho_0 U}{\ell} \frac{D\rho^*}{Dt^*} + \frac{\rho_0 U}{\ell} \rho^* \nabla^* \cdot \mathbf{u}^* = 0 \implies \frac{D\rho^*}{Dt^*} + \rho^* \nabla^* \cdot \mathbf{u}^* = 0. \quad (5.2.13)$$

Similarly non-dimensionalizing the momentum equation, we obtain

$$\frac{\rho_0 U^2}{\ell} \rho^* \left(\frac{D\mathbf{u}^*}{Dt^*} \right) = -\frac{p_0}{\ell} \nabla^* p^* + \frac{\mu U}{\ell^2} \left(2\nabla^* \cdot \mathbf{S}^* - \frac{2}{3} \nabla^* (\nabla^* \cdot \mathbf{u}^*) \right) + \rho_0 g \rho^* \mathbf{g}^*, \quad (5.2.14)$$

where $\mathbf{g}^* = \mathbf{g}/g$. Dividing by $\rho_0 U^2/\ell$, we get,

$$\rho^* \frac{D\mathbf{u}^*}{Dt^*} = \frac{\rho_0}{\Delta\rho} (-\nabla^* p^* + \rho^* \mathbf{g}^*) + \frac{1}{\text{Re}} \left(2\nabla^* \cdot \mathbf{S}^* - \frac{2}{3} \nabla^* (\nabla^* \cdot \mathbf{u}^*) \right). \quad (5.2.15)$$

Note that $\frac{D}{Dt^*} = \frac{\partial}{\partial t^*} + \mathbf{u}^* \cdot \nabla^*$. Also, we have deliberately not used the incompressible momentum equation since it is not known apriori if compressibility may

have an effect or not.

The primary assumption behind Boussinesq approximation is that the density variations in the flow are very small compared to some average density. For instance, ρ_0 may be considered as a time averaged density value at some location, or spatially averaged density value. For simplicity, we consider density as

$$\rho = \rho_0 + \rho', \quad (5.2.16)$$

where ρ_0 is constant and ρ' is the additional variation which is causing the flow. The above expression can be non-dimensionalized as,

$$\rho^* = 1 + \frac{\Delta\rho}{\rho_0} \rho'^*, \quad (5.2.17)$$

where $\rho'^* \sim \mathcal{O}(1)$. Similarly, we consider pressure as

$$p = \bar{p} + p', \implies p^* = \bar{p}^* + \frac{\Delta\rho}{\rho_0} p'^*. \quad (5.2.18)$$

The reason to include $\frac{\Delta\rho}{\rho_0}$ in front of p'^* is that the pressure deviations from hydrostatic values will be caused by flow, which itself is taking place due to velocity, which scales with $\sqrt{\Delta\rho/\rho_0}$. Finally, the velocity can be taken as,

$$\mathbf{u}^* = \frac{\mathbf{u}}{U} = \sqrt{\frac{\Delta\rho}{\rho_0}} \frac{\mathbf{u}}{\sqrt{g\ell}}. \quad (5.2.19)$$

Note that $|\mathbf{u}^*| \sim \mathcal{O}(1)$.

Writing $\frac{\Delta\rho}{\rho_0} = \epsilon$, we obtain,

$$\epsilon \left(\frac{\partial \rho'^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla \rho'^* + \rho'^* \nabla^* \cdot \mathbf{u}^* \right) + \nabla^* \cdot \mathbf{u}^* = 0, \quad (5.2.20)$$

and

$$\epsilon \left(\frac{D\mathbf{u}^*}{Dt^*} - \frac{1}{\text{Re}} \left(2\nabla^* \cdot \mathbf{S}^* - \frac{2}{3} \nabla^* (\nabla^* \cdot \mathbf{u}^*) \right) \right) + \mathcal{O}(\epsilon^2) = -\nabla \bar{p}^* + \mathbf{g}^* + \epsilon (-\nabla p'^* + \rho'^* \mathbf{g}^*). \quad (5.2.21)$$

The reason to introduce ϵ is due to its standard use as *an arbitrary small quantity* ($\epsilon \ll 1$) in asymptotics. We will make extensive use of ϵ in this course, denoting different *small parameters* in different problems. By grouping together terms of

$\epsilon^0, \epsilon^1, \epsilon^2 \dots$, we obtain a hierarchy of equations. Usually these are read as *equation at $\mathcal{O}(\epsilon^n)$* . For instance, continuity at $\mathcal{O}(1)$ is,

$$\nabla^* \cdot \mathbf{u}^* = 0, \quad (5.2.22)$$

and at $\mathcal{O}(\epsilon)$ is

$$\frac{\partial \rho'^*}{\partial t^*} + \nabla \cdot (\rho'^* \mathbf{u}^*) = 0. \quad (5.2.23)$$

Equations obtained at lowest power of ϵ are called as *leading order equations*. The leading order continuity equation yields $\nabla \cdot \mathbf{u}^* = 0$. From momentum equation, we obtain,

$$-\nabla \bar{p}^* + \mathbf{g}^* = 0 \quad (5.2.24)$$

at leading order, and

$$\frac{\partial \mathbf{u}^*}{\partial t} + \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* = -\nabla p'^* + \rho'^* \mathbf{g}^* + \frac{1}{\text{Re}} \nabla^{*2} \mathbf{u}^*. \quad (5.2.25)$$

at $\mathcal{O}(\epsilon)$. Note that we have already used $\nabla^* \cdot \mathbf{u}^* = 0$ in the above momentum equation. From leading order momentum equation, we obtain the hydrostatic variation of pressure in the flow. Since $\mathbf{g}^* = -\hat{k}$, we obtain,

$$\bar{p}^* = -z + \text{const.} \quad (5.2.26)$$

The $\mathcal{O}(\epsilon)$ equation yields the dynamical equation for flow due to bouyancy caused by small density changes $\frac{\Delta \rho}{\rho_0} \ll 1$. Hence, the Boussinesq system of equations become,

$$\nabla^* \cdot \mathbf{u}^* = 0, \quad (5.2.27)$$

$$\frac{D \mathbf{u}^*}{Dt} = -\nabla p' + \rho'^* \mathbf{g}^* + \frac{1}{\text{Re}} \nabla^{*2} \mathbf{u}^*, \quad (5.2.28)$$

$$\frac{\partial \rho'^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla \rho'^* = 0. \quad (5.2.29)$$

The density variations simply get transported as passive-scalars in the flow. To leading order, the flow is incompressible. It is also important to note that we have not assumed It is important to note that the choosing the correct velocity scale in the above problem was essential to obtain this set of equations. Usually, if correct scales are not chosen and asymptotic approximation is applied, funny results are obtained!

Energy equation in Boussinesq approximation

Equations (5.2.27), (5.2.28), and (5.2.29) were obtained only from density variation point of view. These density variations can be caused by any possible physical mechanism such as temperature changes, concentration changes, etc.

When the density changes are coupled with the temperature changes, then density perturbations are related to temperature perturbations. Suppose the relevant scale of temperature is T_0 and the temperature field is,

$$T = T_0 + T', \quad (5.2.30)$$

and the dimensionless temperature is,

$$T^* = 1 + \frac{\Delta T}{T_0} T'^*. \quad (5.2.31)$$

The dimensional temperature and density changes are related to each other via the *thermal expansion coefficient* as,

$$\frac{\rho'}{\rho_0} = -\alpha T', \quad (5.2.32)$$

hence the temperature and density change scales are also related to each other as,

$$\Delta \rho = \rho_0 \alpha \Delta T \implies T^* = 1 + \frac{\epsilon}{\alpha T_0} T'^*. \quad (5.2.33)$$

In these cases, instead of solving the density transport equation, temperature transport equation has to be solved which can be derived using the internal energy Eq. (3.5.17). Ignoring any separate source terms and writing $e_I = c_v T$,

$$\rho c_v \left(\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T \right) = -p \nabla \cdot \mathbf{u} + k \nabla^2 T + \phi. \quad (5.2.34)$$

With a recourse to thermodynamics, we highlight that the heat capacity at constant volume and constant pressure are exactly equal for a purely incompressible fluids (with constant density). However, if the density varies with temperature,

then

$$\begin{aligned} p \nabla \cdot \mathbf{u} &= -\frac{p}{\rho} \frac{D\rho}{Dt} = -\frac{p}{\rho} \left(\left(\frac{\partial \rho}{\partial T} \right)_p \frac{DT}{Dt} + \left(\frac{\partial \rho}{\partial p} \right)_T \frac{Dp}{Dt} \right) \\ &= p\alpha \frac{DT}{Dt} - \frac{p}{\rho} \left(\frac{\partial \rho}{\partial p} \right)_T \frac{Dp}{Dt}, \end{aligned} \quad (5.2.35)$$

using which, we can write,

$$(\rho c_v + \alpha p) \left(\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T \right) = k \nabla^2 T + \frac{p}{\rho} \left(\frac{\partial \rho}{\partial p} \right)_T \frac{Dp}{Dt} + \phi. \quad (5.2.36)$$

For gases, αp can be computed using the equation of state. For instance, for an ideal gas, $\alpha = 1/T$, hence $\alpha p = \rho R$ which yields $\rho c_v + \rho R = \rho c_p$. Furthermore, we will assume that $\left(\frac{\partial \rho}{\partial p} \right)_T$ is very small. Usually, this can be reasoned with the help of another dimensionless number called the Mach number, which introduces another smallness parameter in the problem (separate from $\Delta\rho/\rho$). Deferring this discussion to compressible flow discussion, we simply proceed with ignoring the $\frac{Dp}{Dt}$ term on the RHS.

For liquids, we use that $\frac{\alpha p}{\rho c_v} \ll 1$, which is not connected to $\Delta\rho/\rho_0 \ll 1$. Hence, this is an additional assumption.

For both liquids and gases, under very small Mach number and small density change assumptions, we obtain,

$$\rho C \frac{DT}{Dt} = k \nabla^2 T + \phi, \quad (5.2.37)$$

where $C = c_p$ for gases. Note that we have assumed k to be constant in the above equation. Additionally, we will assume constant C throughout.

Non-dimensionalizing the equation, we obtain,

$$\frac{\rho_0 C T_0 U}{\ell} \rho^* \frac{DT^*}{Dt^*} = \frac{k T_0}{\ell^2} \nabla^{*2} T^* + \frac{\mu U^2}{\ell^2} \phi^*, \quad (5.2.38)$$

which yields,

$$\epsilon (1 + \epsilon \rho'^*) \frac{DT'^*}{Dt'^*} = \epsilon \frac{k}{\rho_0 C \ell U} \nabla^{*2} T'^* + \frac{\mu U^2}{\rho_0 C T_0 \ell U} \phi^*. \quad (5.2.39)$$

Here we identify two more dimensionless parameters, one of which we have already

assumed to be small. We write,

$$\frac{k}{\rho_0 C \ell U} = \frac{1}{\text{RePr}}, \quad (5.2.40)$$

where Pr is the Prandtl number defined as ,

$$\text{Pr} = \frac{\mu C}{k} = \frac{\nu}{\kappa}. \quad (5.2.41)$$

Additionally, we identify,

$$\frac{\mu U^2}{\rho_0 C T_0 \ell U} = \frac{M^2}{\text{Re}}, \quad (5.2.42)$$

where we have written $\frac{U^2}{C T_0} = M^2$ which is the Mach number. Note that this differs from the usual definition of the Mach number (flow speed to speed of sound) by a constant factor. Here M^2 can be seen as the ratio of flow kinetic energy to its internal energy/enthalpy. We state that $\frac{M^2}{\text{Re}} \ll \epsilon \ll 1$ and hence, the temperature equation at $\mathcal{O}(\epsilon)$ which holds for us is,

$$\frac{DT'^*}{Dt^*} = \frac{1}{\text{RePr}} \nabla^{*2} T'^* \quad (5.2.43)$$

We note that $\rho'^* = -T'^*$, which yields the equation for density fluctuations,

$$\frac{D\rho'^*}{Dt^*} = \frac{1}{\text{RePr}} \nabla^{*2} \rho'^*, \quad (5.2.44)$$

which modifies the density transport equation with a diffusive flux on account of thermal diffusion. There is an ambiguity now! Should we use Eq. (5.2.44) or the previously derived equation with no density diffusion. The catch is, in previous derivation, we did not account for the diffusion of density. Diffusion of density results in kinematic changes, since the velocity $\mathbf{u}$ we solve is the velocity of the center of mass of a fluid particle. For a stationary fluid particle with diffusing density gradients, the center of mass will shift resulting in an additional diffusive velocity.

There are many other dimensionless numbers which appear in various flow problems. It is important to note that the boundary and interface conditions should also be non-dimensionalized to obtain a full set of dimensionless equations and all relevant dimensionless numbers. In deciding the relevant scales, we used the standard Π grouping taught in introductory fluid mechanics classes. Below we briefly review the Buckingham's Pi theorem and its applicability in presenting

experimental data and similarity.

5.3 Pi Theorem

Let n quantities $q_1, q_2, q_3 \cdots q_n$ be measured in an experiment. If these quantities depend on each other, then we can write an implicit equation

$$f(q_1, q_2, q_3 \cdots q_n) = 0. \quad (5.3.1)$$

According to the Buckingham's Pi theorem, this relation can be made more concise. Let us denote the dimension matrix of these n quantities as ,

q_1	q_2	q_3	q_4	q_5	q_n	
M	1	1	0	0	0	1
L	3	0	-2	0	0	1
T	0	1	2	0	0	1
K	0	1	-2	1	0	0

$$. \quad (5.3.2)$$

Let r be the rank of the rectangular matrix. In such a case, the relation in Eq. (5.3.1) can be reduced to $n-r$ dimensionless variables. The rank r essentially tells which exponents of which quantities make an independent set and span all the dimensions. The rest of the $n-r$ variables can be non-dimensionalized to form the so-called Π groups such that,

$$g(\Pi_1, \Pi_2, \Pi_3, \cdots \Pi_n) = 0. \quad (5.3.3)$$

For instance (see ?), if pressure drop in a pipe flow Δp depends on diameter d , length l , roughness e , speed U , density ρ , and viscosity μ , then

$$f(\Delta p, d, l, e, U, \rho, \mu) = 0. \quad (5.3.4)$$

The dimensional matrix of this can be formulated as,

	Δp	d	l	e	U	ρ	μ
M	1	0	0	0	0	1	1
L	-1	1	1	1	1	-3	-1
T	-2	0	0	0	-1	0	-1

$$. \quad (5.3.5)$$

The rank of this matrix is 3 and the independent variables can be formed using U , ρ , μ . Even U , ρ , and d can be used as independent variables. Using these three, we obtain the following Π groups,

$$g\left(\frac{\Delta p}{\rho U^2}, \frac{l}{d}, \frac{e}{d}, \frac{\mu}{\rho U d}\right) = 0. \quad (5.3.6)$$

Another example which demonstrates the use of formulating the dimensional matrix is that of the capillary effect due to which fluid rises in a small tube dipped through the surface. Let the height rise h be dependent upon the diameter of the tube d , surface tension coefficient σ , and the specific weight γ which is density times gravity of the liquid. The dimensional matrix looks like,

$$\begin{array}{c|cccc} & h & d & \gamma & \sigma \\ \hline M & 0 & 0 & 1 & 1 \\ L & 1 & 1 & -2 & 0 \\ T & 0 & 0 & -2 & -2 \end{array}. \quad (5.3.7)$$

Note how the bottom most row is simply -2 times the first row. Hence, naturally, the rank of this matrix can not be 3. Inspection shows that the rank is 2, which means that only 2 independent variables can be picked. Since h is the quantity of interest, we pick d and γ , which yields,

$$g\left(\frac{\sigma}{d^2 \gamma}, \frac{h}{d}\right) = 0. \quad (5.3.8)$$

Note that this process automatically yields the *characteristic scales* which we discussed in the previous section. However, as in every problem, a variety of characteristic scales are present. For instance, both ρU^2 and $\mu U/d$ are valid pressure scales. The choice requires physical intuition of the flow problem.