

Chapter 6

Low Reynolds number flows

In this chapter, we will look at the low Reynolds number flows, also often called the creeping flows or Stokes flows. We will formulate the governing equation, and solve the drag on a sphere in a low Reynolds number flow. We will also show the usage in lubrication theory.

6.1 Correct pressure scale and Stokes equation

To outline the correct governing equations for Stokes flows, we note that the pressure scale p_0 in Eq. (5.2.4) needs to be chosen appropriately. For instance, if $p_0 = \rho U^2$ is chosen, then the momentum equation reduces to,

$$\nabla^{*2} \mathbf{u}^* - \text{Re} \left(\nabla^* p^* + \frac{D\mathbf{u}^*}{Dt^*} \right) = 0. \quad (6.1.1)$$

If we pick $\text{Re} \ll 1$ and consider the leading order equation, we obtain,

$$\nabla^{*2} \mathbf{u}^* = 0. \quad (6.1.2)$$

An interesting property of the Laplacian is that the maximum/minimum of the solution can only exist on the boundary. So, if the flow is bounded by walls, the only possible solution will be no-flow! This means, one can not have a low Re flow in a pipe or a channel. Which is certainly not true given that we can maintain a very small pressure difference across the ends of a pipe and drive a flow slowly.

This happened because we considered the wrong scale for pressure. The correct scale for pressure in a flow where viscous stresses dominate the inertial stress ρU^2 ,

will be

$$p_0 = \frac{\mu U}{\ell}, \quad (6.1.3)$$

using which we obtain,

$$\nabla^{*2} \mathbf{u}^* - \nabla^* p^* - \text{Re} \left(\frac{D\mathbf{u}^*}{Dt^*} \right) = 0. \quad (6.1.4)$$

Considering the asymptotic expansions,

$$\mathbf{u}^* = \mathbf{u}^{(0)} + \text{Re} \mathbf{u}^{(1)} + \text{Re}^2 \mathbf{u}^{(2)} + \dots, \quad (6.1.5)$$

$$p^* = p^{(0)} + \text{Re} p^{(1)} + \text{Re}^2 p^{(2)} + \dots, \quad (6.1.6)$$

using which, we obtain the leading order equation as (dropping the cumbersome $()^{(0)}$ notation,

$$\nabla^{*2} \mathbf{u}^* = \nabla^* p^*, \quad (6.1.7)$$

which in dimensional form reads,

$$\nabla p = \mu \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0. \quad (6.1.8)$$

The above is called the steady Stokes equation. It describes a steady low Reynolds number flow up to leading order. This means that if compared against experimental data, the analytical solution will consist of errors of $\mathcal{O}(\text{Re})$. Note that the above equation was obtained assuming a time scale of ℓ/U in the flow. In some situations, there may be an externally imposed time scale which can result in the so-called unsteady Stokes equations,

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\nabla p}{\rho} = \mu \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0. \quad (6.1.9)$$

Steady Stokes flows satisfy linear equations. Vorticity is governed by

$$\nabla \times (\nabla p = \mu \nabla^2 \mathbf{u}) \implies \nabla^2 \boldsymbol{\omega} = 0, \quad (6.1.10)$$

and pressure is governed by,

$$\nabla \cdot (\nabla p = \mu \nabla^2 \mathbf{u}) \implies \nabla^2 p = 0. \quad (6.1.11)$$

Since all equations are linear, flow fields can be linearly superimposed to produce new flow fields. We also note that if we reverse velocity $\mathbf{u} \rightarrow -\mathbf{u}$, the governing equations remain the same with $p - p_0 \rightarrow -(p - p_0)$. Hence, Stokes flows are

reversible in the sense that reverse of a Stokes flow is a Stokes flow. Furthermore, Stokes flow past a symmetric object is symmetric.

6.2 Stokes stream function

The Stokes stream function is defined for flows which may be planar flows or axisymmetric flows. Let there be orthogonal coordinates ξ_1, ξ_2, ξ_3 with unit vectors $\hat{\mathbf{e}}_i$, and the corresponding metric coefficients h_1, h_2, h_3 . An incremental displacement $d\mathbf{r}$ is

$$d\mathbf{r} = \sum_i \frac{d\xi_i}{h_i} \hat{\mathbf{e}}_i. \quad (6.2.1)$$

Hence, gradient of a scalar is defined as,

$$\nabla\phi = \sum_i h_i \hat{\mathbf{e}}_i \frac{\partial\phi}{\partial\xi_i}. \quad (6.2.2)$$

Consider a vector $\mathbf{A} = \hat{\mathbf{e}}_3 A(\xi_1, \xi_2)$. For such a vector, we get the curl defined as (considering orthogonal curvilinear coordinates, see Chapter 1),

$$\nabla \times \mathbf{A} = \hat{\mathbf{e}}_1 h_2 h_3 \frac{\partial}{\partial\xi_2} \left(\frac{A}{h_3} \right) - \hat{\mathbf{e}}_2 h_3 h_1 \frac{\partial}{\partial\xi_1} \left(\frac{A}{h_3} \right). \quad (6.2.3)$$

The reason to consider $\nabla \times \mathbf{A}$ is that by definition $\nabla \cdot (\nabla \times \mathbf{A}) = 0$. Hence, we can construct an incompressible velocity field which is only a function of ξ_1 and ξ_2 coordinates (uniform in ξ_3) and also has components in only 1 and 2 directions. We can define the curl of curl similarly as,

$$\nabla \times (\nabla \times \mathbf{A}) = \hat{\mathbf{e}}_3 h_1 h_2 \left[-\frac{\partial}{\partial\xi_1} \left(\frac{h_3 h_1}{h_2} \frac{\partial}{\partial\xi_1} \left(\frac{A}{h_3} \right) \right) - \frac{\partial}{\partial\xi_2} \left(\frac{h_2 h_3}{h_1} \frac{\partial}{\partial\xi_2} \left(\frac{A}{h_3} \right) \right) \right]. \quad (6.2.4)$$

As intimidating these expressions are, these can be derived easily once the generalized curvilinear coordinate definitions are understood properly.

The purpose of the above discussion is to show that for symmetric flows are interested in, we can construct a velocity field which satisfies the continuity equation $\nabla \cdot \mathbf{u} = 0$ such that,

$$\mathbf{u} = \nabla \times \mathbf{A}, \text{ where } \mathbf{A} = \hat{\mathbf{e}}_3 h_3 \psi(\xi_1, \xi_2), \quad (6.2.5)$$

and ψ is called the Stokes streamfunction. Notice that it doesn't need the flow to be Stokes flow for defining the streamfunction. Conversely, this is useful in

solving Stokes flow problems. Given above definition, we obtain that

$$\mathbf{u} = \hat{\mathbf{e}}_1 h_2 h_3 \frac{\partial \psi}{\partial \xi_2} - \hat{\mathbf{e}}_2 h_1 h_3 \frac{\partial \psi}{\partial \xi_1}. \quad (6.2.6)$$

Additionally, we get the vorticity as,

$$\boldsymbol{\omega} = \nabla \times (\nabla \times \mathbf{A}) = -\hat{\mathbf{e}}_3 h_1 h_2 \left[\frac{\partial}{\partial \xi_1} \left(\frac{h_3 h_1}{h_2} \frac{\partial \psi}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left(\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial \xi_2} \right) \right]. \quad (6.2.7)$$

Let us define an operator called E^2 operator such that,

$$h_3 E^2() = h_1 h_2 \left[\frac{\partial}{\partial \xi_1} \left(\frac{h_3 h_1}{h_2} \frac{\partial ()}{\partial \xi_1} \right) + \frac{\partial}{\partial \xi_2} \left(\frac{h_2 h_3}{h_1} \frac{\partial ()}{\partial \xi_2} \right) \right]. \quad (6.2.8)$$

Then $\boldsymbol{\omega} = -\hat{\mathbf{e}}_3 h_3 E^2(\psi)$. Finally, $\nabla^2 \boldsymbol{\omega} = -\nabla \times (\nabla \times \boldsymbol{\omega})$ and we get the equation for ψ for Stokes flows as,

$$\nabla^2 \boldsymbol{\omega} = 0 \implies -\hat{\mathbf{e}}_3 h_3 E^2(E^2(\psi)) = 0. \quad (6.2.9)$$

6.2.1 2D planar Stokes flows

In cartesian coordinates $\xi_1 = x, \xi_2 = y$, and $h_1 = h_2 = h_3 = 1$, we see that,

$$\mathbf{A} = \hat{k} \psi(x, y) \implies \mathbf{u} = \hat{i} \frac{\partial \psi}{\partial x} - \hat{j} \frac{\partial \psi}{\partial y}, \quad (6.2.10)$$

which satisfies the continuity equation by definition. In polar coordinates with $\xi_1 = r, \xi_2 = \theta$, and $h_1 = 1, h_2 = \frac{1}{r}, h_3 = 1$,

$$\mathbf{A} = \hat{k} \psi(r, \theta) \implies \mathbf{u} = \hat{\mathbf{e}}_r \frac{1}{r} \frac{\partial \psi}{\partial \theta} - \hat{\mathbf{e}}_\theta \frac{\partial \psi}{\partial r}. \quad (6.2.11)$$

The vorticity is then given by $\boldsymbol{\omega} = -\nabla^2 \psi \hat{k}$ and the streamfunction satisfies the so-called biharmonic governing equation,

$$\nabla^2 (\nabla^2 \psi) = \nabla^4 \psi = 0. \quad (6.2.12)$$

6.2.2 Axisymmetric flows in spherical coordinates

In spherical coordinates r, θ, φ , the metric coefficients are $1, \frac{1}{r}, \frac{1}{r \sin \theta}$. The stream function can be picked as,

$$\mathbf{A} = \hat{\mathbf{e}}_\varphi \frac{\psi(r, \theta)}{r \sin \theta} \implies \mathbf{u} = \hat{\mathbf{e}}_r \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} - \hat{\mathbf{e}}_\theta \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}. \quad (6.2.13)$$

This $\mathbf{u}$ automatically satisfies $\nabla \cdot \mathbf{u} = 0$ in spherical coordinates. The E^2 operator becomes

$$E^2() = \frac{\partial^2()}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial()}{\partial \theta} \right). \quad (6.2.14)$$

It is important to note that ,

$$\nabla^2() = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial()}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial()}{\partial \theta} \right) \neq E^2(). \quad (6.2.15)$$

The streamfunction satisfies the equation

$$E^2(E^2\psi) = 0. \quad (6.2.16)$$

6.3 Stokes flow past a sphere

Consider uniform flow of velocity U_0 approaching a sphere of radius R . We can align a circular ring on the sphere such that its axis points in the direction of the flow. Then, the flow is uniform in the tangential direction of that ring, which we identify as the azimuthal direction ϕ here. The governing equation is give by

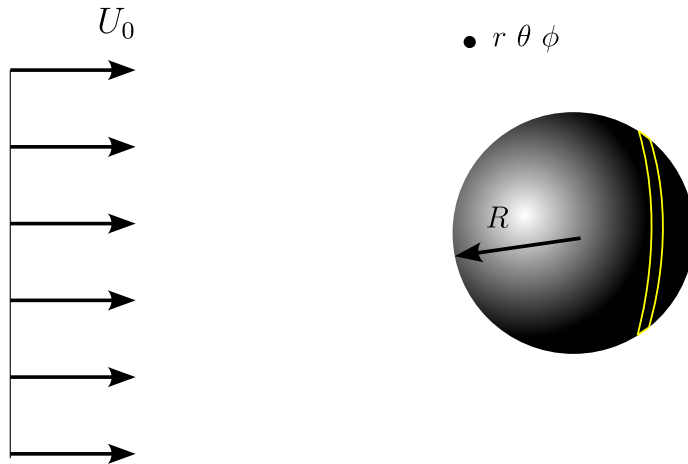


Figure 6.1: Stokes flow past a sphere

$$E^2(E^2\psi) = 0, \quad (6.3.1)$$

where E^2 is given by Eq. (6.2.14). The boundary conditions we use are,

$$\mathbf{u}(r = R) = 0 \quad (6.3.2)$$

due to no-slip and hard wall. Hence $u_r = 0, u_\theta = 0$. Substituting in terms of streamfunction,

$$\frac{\partial\psi}{\partial\theta} = 0, \quad \frac{\partial\psi}{\partial r} = 0, \quad \text{at } r = R. \quad (6.3.3)$$

Furthermore, at $r \rightarrow \infty$, we pose that the flow changes *negligibly* and remains the uniform flow $U_0\hat{i}$. Hence,

$$\frac{1}{r^2 \sin \theta} \frac{\partial\psi}{\partial\theta} = U_0 \cos \theta, \quad \frac{1}{r \sin \theta} \frac{\partial\psi}{\partial r} = U_0 \sin \theta, \quad \text{at } r \rightarrow \infty. \quad (6.3.4)$$

Since this limit is applied to $r \rightarrow \infty$, from $\frac{\partial\psi}{\partial\theta}$ we see that

$$\psi = g(r) \sin^2 \theta, \quad g(r) = \frac{U_0 r^2}{2} \text{ for } r \rightarrow \infty. \quad (6.3.5)$$

Substituting in the governing equation, we obtain that

$$E^2\psi = \left(g'' - \frac{2g}{r^2}\right) \sin^2 \theta, \quad (6.3.6)$$

from which we get,

$$E^2(E^2\psi) = g'''' - \frac{4g''}{r^2} + \frac{8g'}{r^3} - \frac{8g}{r^4} = 0. \quad (6.3.7)$$

The above is an Euler-Cauchy equation with a general solution,

$$g(r) = \frac{A}{r} + Br + Cr^2 + Dr^4. \quad (6.3.8)$$

For $\lim_{r \rightarrow \infty}$ we note that $g(r) \rightarrow \frac{U_0 r^2}{2}$. Hence $D = 0$ and $C = \frac{U_0}{2}$ leaving,

$$g(r) = \frac{A}{r} + Br + \frac{U_0 r^2}{2}. \quad (6.3.9)$$

Substituting the boundary conditions at $r = R$, we obtain $A = U_0 R^3/4$ and $B = -3U_0 R/4$. This completes the solution for $\psi(r, \theta)$ as,

$$\psi = U_0 R^2 \left(\frac{R}{4r} - \frac{3r}{4R} + \frac{r^2}{2R^2} \right) \sin^2 \theta. \quad (6.3.10)$$

This also yields the velocity components

$$u_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} = U_0 \cos \theta \left(1 + \frac{R^3}{2r^3} - \frac{3R}{2r} \right), \quad (6.3.11)$$

$$u_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} = -U_0 \sin \theta \left(1 - \frac{R^3}{4r^3} - \frac{3R}{4r} \right). \quad (6.3.12)$$

To evaluate the drag force, we need to evaluate the pressure and the viscous stress. We use the Stokes equation $\nabla p = \mu \nabla^2 \mathbf{u}$. To compute $\nabla^2 \mathbf{u} = -\nabla \times \boldsymbol{\omega}$ we compute vorticity since its informative as well,

$$\begin{aligned} \boldsymbol{\omega} &= -\frac{\hat{\mathbf{e}}_\phi}{r} \left(\frac{\partial}{\partial r} \left(\frac{1}{\sin \theta} \frac{\partial \psi}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \right) \right) \\ &= -\frac{3U_0 R}{2r^2} \sin \theta \hat{\mathbf{e}}_\phi. \end{aligned} \quad (6.3.13)$$

Now computing the pressure gradients, we obtain,

$$\frac{\partial p}{\partial r} = -\mu (\nabla \times \boldsymbol{\omega}) \cdot \hat{\mathbf{e}}_r = \frac{3U_0 R \mu}{r^3} \cos \theta, \quad (6.3.14)$$

$$\frac{1}{r} \frac{\partial p}{\partial \theta} = -\mu (\nabla \times \boldsymbol{\omega}) \cdot \hat{\mathbf{e}}_\theta = \frac{3\mu U_0 R}{2r^3} \sin \theta. \quad (6.3.15)$$

To obtain pressure on the surface of the sphere $p(R, \theta)$, we use the above partial derivatives

$$p - p_\infty = \int_\infty^R \frac{\partial p}{\partial r} dr + \int_0^\theta \frac{\partial p}{\partial \theta} d\theta = -\frac{3\mu U_0}{2R} \cos \theta \quad (6.3.16)$$

Surface force due to pressure is given by

$$\mathbf{F}_p = - \int (p - p_\infty) \hat{\mathbf{n}} dS. \quad (6.3.17)$$

Since $\hat{\mathbf{n}} = \hat{\mathbf{e}}_r$, $dS = R^2 \sin \theta d\phi d\theta$, we obtain,

$$\mathbf{F}_p = \frac{3\mu U_0}{2R} \int_0^\pi \int_0^{2\pi} \hat{\mathbf{e}}_r \cos \theta R^2 \sin \theta d\phi d\theta = \frac{3\mu U_0 R}{2} \int_0^\pi \int_0^{2\pi} \hat{\mathbf{e}}_r \cos \theta \sin \theta d\phi d\theta. \quad (6.3.18)$$

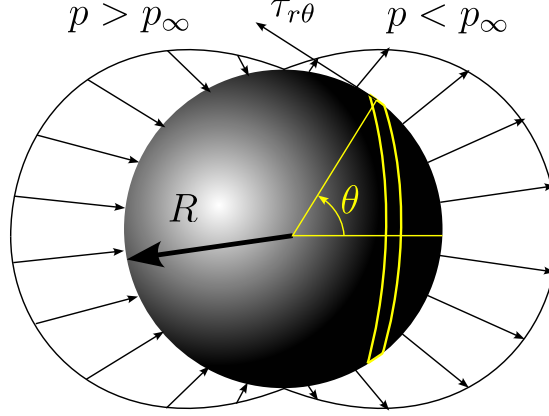


Figure 6.2: Pressure and surface viscous stress on sphere in a uniform Stokes flow.

Note that $\hat{\mathbf{e}}_r = \cos \theta \hat{\mathbf{i}} + \sin \theta \cos \phi \hat{\mathbf{j}} + \sin \theta \sin \phi \hat{\mathbf{k}}$ if x is oriented along the flow, y is normal in the plane of the paper, and z is outside the paper. When we integrate $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ components over ϕ , they vanish and only $\hat{\mathbf{i}}$ component survives such that,

$$\mathbf{F}_p = 3\mu\pi U_0 R \int_0^\pi \cos^2 \theta \sin \theta d\theta \hat{\mathbf{i}} = 2\pi\mu U_0 R \hat{\mathbf{i}}. \quad (6.3.19)$$

It is important to note that only flow kinematics are symmetric. The pressure is not symmetric upstream and downstream of the sphere. If that was the case, then there would not have been any pressure force on the sphere. To obtain the viscous forces, we compute the viscous stresses on the sphere. Noting that the flow is axisymmetric, we expect only τ_{rr} , $\tau_{r\theta}$, and $\tau_{\theta\theta}$ to exist (notice that I have replaced 1, 2, 3 indices with the usual spherical coordinates). We note,

$$\begin{aligned} \tau_{rr} &= 2\mu \frac{\partial u_r}{\partial r} = U_0 \cos \theta \left(\frac{3R}{2r^2} - \frac{3R^3}{2r^4} \right) = 0 \text{ at } r = R, \\ \tau_{r\theta} &= \tau_{\theta r} = \mu \left(r \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right) + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) = -\frac{3\mu U_0}{2R} \sin \theta \text{ at } r = R, \\ \tau_{\theta\theta} &= 2\mu \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) = 0 \text{ at } r = R. \end{aligned} \quad (6.3.20)$$

The viscous force $\mathbf{F}_v$ can be calculated as,

$$\mathbf{F}_v = \int_S \boldsymbol{\tau} \cdot \hat{\mathbf{n}} dS = \int_S \tau_{r\theta} \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_r dS = \int_S \tau_{r\theta} \hat{\mathbf{e}}_\theta dS. \quad (6.3.21)$$

Noting that $\hat{\mathbf{e}}_\theta = \sin \theta \hat{\mathbf{i}} + \cos \theta \cos \phi \hat{\mathbf{j}} + \cos \theta \sin \phi \hat{\mathbf{k}}$. Again, we note that integrals

over ϕ vanish due to axisymmetry, with the following expression for viscous force,

$$\mathbf{F}_v = 3\pi\mu U_0 R \int_0^\pi \sin^2 \theta \cos \theta d\theta \hat{i} = 4\pi\mu U_0 R \hat{i}. \quad (6.3.22)$$

Combining the above, we obtain the total drag force on the sphere as $6\pi\mu U_0 R \hat{i}$.

6.3.1 Problems at $r/R \gg 1$

From the solution, let us calculate $\mathbf{u} \cdot \nabla \mathbf{u}$, which is the inertial/convective term that we ignored. Why we are calculating the term which was initially reasoned to be negligible will become clear. For axisymmetric spherical flows,

$$\mathbf{u} \cdot \nabla \mathbf{u} = \left(\mathbf{u} \cdot \nabla u_r - \frac{u_\theta^2}{r} \right) \hat{e}_r + \left(\mathbf{u} \cdot \nabla u_\theta + \frac{u_r u_\theta}{r} \right) \hat{e}_\theta, \quad (6.3.23)$$

where

$$\mathbf{u} \cdot \nabla = u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta}. \quad (6.3.24)$$

While the above expression becomes lengthy, looking only at $\hat{e}_r$ component is informative since,

$$(\mathbf{u} \cdot \nabla \mathbf{u})_r = \frac{3U_0^2 R \cos^2 \theta}{r^2} \left(\frac{1}{2} - \frac{1}{2} \frac{R^2}{r^2} \right) \left(1 + \frac{1}{2} \frac{R^3}{r^3} - 3 \frac{R}{r} \right) + \frac{U_0^2 \sin^2 \theta R}{r^2} \left(\frac{3R}{4r} + \mathcal{O} \left(\frac{R}{r} \right)^2 \right). \quad (6.3.25)$$

For very small R/r or very large r/R , we note that the above expression at leading order has the following scale

$$[(\mathbf{u} \cdot \nabla \mathbf{u})_r] = \frac{U_0^2 R}{r^2}. \quad (6.3.26)$$

Similarly, one can confirm the same for the θ component. Now, looking at the ratio of the scales of this inertial term and the viscous term (evaluated above) we obtain,

$$\frac{[\rho \mathbf{u} \cdot \nabla \mathbf{u}]}{[\mu \nabla^2 \mathbf{u}]} = \frac{\rho r U_0}{\mu} = \frac{\rho R U_0}{\mu} \frac{r}{R} = \text{Re}_R \frac{r}{R}. \quad (6.3.27)$$

The above equation shows that when $\frac{r}{R} \sim \frac{1}{\text{Re}_R} \gg 1$, the inertial terms and the viscous terms have similar scales and one can not be neglected compared to the other. The above discussion highlights, that the Stokes solution is a *uniform* approximation at $\mathcal{O}(1)$, just like we set out to derive. And only $\mathcal{O}(1)$ terms from ψ should be used to assess the validity of the assumption that the inertial

terms are small compared to the viscous terms. Intuitively, this can be seen as follows - as we move very far away from the sphere, the effect of viscosity due to flow deflection by the sphere is as weak as the inertial terms. For instance, substituting $\xi = r\text{Re}_R$, we obtain,

$$\psi = U_0 R^2 \left(\text{Re}_R \frac{R}{4\xi} - \frac{1}{\text{Re}_R} \frac{3\xi}{4R} + \frac{1}{\text{Re}_R^2} \frac{\xi^2}{2R^2} \right) = \frac{U_0 R^2}{\text{Re}_R^2} \left(\frac{\xi^2}{2R^2} + \text{Re}_R^3 \frac{R}{4\xi} - \text{Re}_R \frac{3\xi}{4R} \right). \quad (6.3.28)$$

The above equation shows that as $\xi \sim 1$, $\text{Re}_R^2 \psi = \frac{U_0^2 \xi^2}{2} + \mathcal{O}(\text{Re}_R)$. In terms of velocity, we see that

$$u_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} = U_0 \cos \theta \left(1 + \frac{R^3}{2r^3} - \frac{3R}{2r} \right) = U_0 \cos \theta \left(1 - \text{Re}_R \frac{3R}{2\xi} + \text{Re}_R^3 \frac{R^3}{2\xi^3} \right), \quad (6.3.29)$$

$$u_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} = -U_0 \sin \theta \left(1 - \frac{R^3}{4r^3} - \frac{3R}{4r} \right) = -U_0 \sin \theta \left(1 - \text{Re}_R^3 \frac{R^3}{4\xi^3} - \text{Re}_R \frac{3R}{4\xi} \right). \quad (6.3.30)$$

For $\xi \sim 1$, the velocity approaches uniform stream with an error of $\mathcal{O}(\text{Re}_R)$. If we recall original asymptotic expansions (which can be cast in terms of ψ) simply as,

$$\psi = \psi_0 + \text{Re}_R \psi_1 + \text{Re}_R^2 \psi_2 + \dots. \quad (6.3.31)$$

Stokes solution gives ψ_0 . To obtain a first order correction, i.e. ψ_1 a different equation must be obtained in the region $r \sim \frac{R}{\text{Re}_R}$.

6.3.2 Oseen's correction - A glimpse of matched asymptotics

In the following discussion, we drop the cumbersome $()^*$ for the dimensionless variables. The equation obtained by Oseen in 1910 (see [Proudman and Pearson \[1957\]](#) for full details) gives us the tool to obtain $\mathcal{O}(\text{Re}_R)$ corrections in the solution. The reasoning goes like this :- The Stokes' solution clearly is not valid for $r \rightarrow \infty$ since the Stokes' equation is not valid. For $\frac{r}{R} \sim \frac{1}{\text{Re}_R}$, finding ψ_1 in the asymptotic expansion implies finding the perturbation in the otherwise uniform flow stream. The perturbation must decay to 0 as $r \rightarrow \infty$. Also, as $r\text{Re}_R \rightarrow 0$, these perturbations must match the perturbations to the uniform stream predicted by the Stokes solution. [Proudman and Pearson \[1957\]](#) suggest the following,

Near field : For $\frac{r\text{Re}_r}{R} \ll 1$

In this region, the Stokes approximations hold and the dimensionless equation is (omitting the $()^*$,

$$\nabla^2 \mathbf{u} - \nabla p - \text{Re}_R \mathbf{u} \cdot \nabla \mathbf{u} = 0. \quad (6.3.32)$$

The above equation can be transformed to an equation for ψ by taking a cross-product and noting $\boldsymbol{\omega} = \nabla \times \mathbf{u} = -E^2 \psi$ for axisymmetric flows. Then, the following Stoke's equation must be used to find solutions,

$$\psi = \psi_0 + \text{Re}_R \psi_1 + \text{Re}_R^2 \psi_2 + \dots \quad (6.3.33)$$

in which ψ_0 is the Stokes' solution that we have found in Eq. (6.3.10). These equations are obtained by taking the characteristic length scale as R , the radius of the sphere.

Far field : For $\frac{\text{Re}_R r}{R} \sim 1$

In this region, the appropriate length scale is $\frac{R}{\text{Re}_R}$. To derive the equation, we consider the dimensionless equation again,

$$\frac{U_0^2}{\ell} (\mathbf{u}^* \cdot \nabla^* \mathbf{u}^*) = -\frac{p_0}{\rho \ell} \nabla^* p^* + \frac{\nu U_0}{\ell^2} \nabla^{*2} \mathbf{u}^*. \quad (6.3.34)$$

If we take $\ell = R$, we obtain the Stokes equation at the leading order. However, if we take $\ell = R/\text{Re}_R$, then,

$$\frac{\nu}{U_0 \ell} = 1, \quad \frac{p_0}{\rho U_0^2} = \frac{\mu U_0}{\ell \rho U_0^2} = 1, \quad (6.3.35)$$

and hence, the dimensionless equation to consider is (omitting $()^*$),

$$\nabla^2 \mathbf{u} - \nabla p - \mathbf{u} \cdot \nabla \mathbf{u} = 0. \quad (6.3.36)$$

The expansion to consider in terms of either $\mathbf{u}, p$ is

$$(\mathbf{u}, p) = (\mathbf{u}_0, p_0) + \text{Re}_R (\mathbf{u}_1, p_1) + \text{Re}_R (\mathbf{u}_2, p_2) + \dots \quad (6.3.37)$$

or ψ as,

$$\psi = \Psi_0 + \text{Re}_R \Psi_1 + \text{Re}_R^2 \Psi_2 + \dots \quad (6.3.38)$$

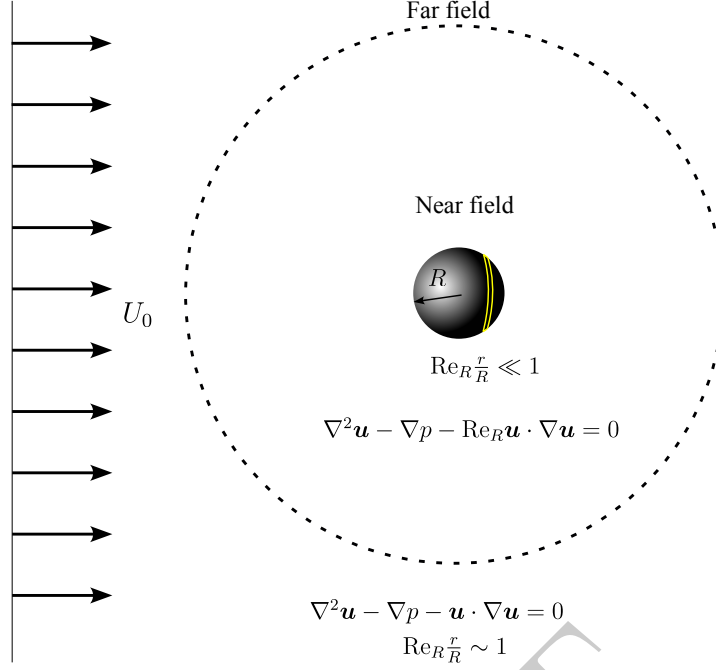


Figure 6.3: Near field and far field equations for Stokes flow past a sphere.

where we have used Ψ to distinguish solution in far field to solution in near field. where $\mathbf{u}_0, p_0$ correspond to the uniform flow stream and p_∞ . At $\mathcal{O}(\text{Re}_R)$, we obtain,

$$\nabla^2 \mathbf{u}_1 - \nabla p_1 - \mathbf{u}_0 \cdot \nabla \mathbf{u}_1 = 0 \quad (6.3.39)$$

which is the Oseen's equation. To solve for $\mathbf{u}_1, p_1$ or correspondingly ψ_1 , we use the asymptotic matching technique. Consider $\xi = r\text{Re}_R/R$. According to this technique $\xi \rightarrow 0$ functions should match the function for $r \rightarrow \infty$. From Eq. (6.3.28) we note that (after non-dimensionalization),

$$\text{Re}_R^2 \psi_0 = \frac{\xi^2}{2} - \text{Re}_R \frac{3\xi}{4} + \text{Re}_R^3 \frac{1}{4\xi}. \quad (6.3.40)$$

Hence,

$$\lim_{r \rightarrow \infty} \text{Re}_R^2 \psi_0(r, \theta) = \lim_{\xi \rightarrow 0} \Psi \quad (6.3.41)$$

What Oseen did was add the $\Psi_0 + \text{Re}_R \Psi_1$ solutions to the Stokes flow solution ψ_0 after removing the common part. This way, the solution holds for large r but has an $\mathcal{O}(\text{Re}_R)$ error in the near field. Technically, the solution must be combined along with ψ_1 . Interested students must follow Proudman and Pearson [1957] for details. The technique of eigenfunction expansion will also become clear after reading the paper.

6.4 Lubrication theory

Another example where low Reynolds number flow assumption is valid is in flow in very thin gaps, or a lubrication problem. The walls enclosing this flow may be moving or stationary. The true essence of the lubrication problem is, what if the length scales in different directions are different? Below, we show how lubrication approximation equations are derived for a planar flow in a channel with varying height.

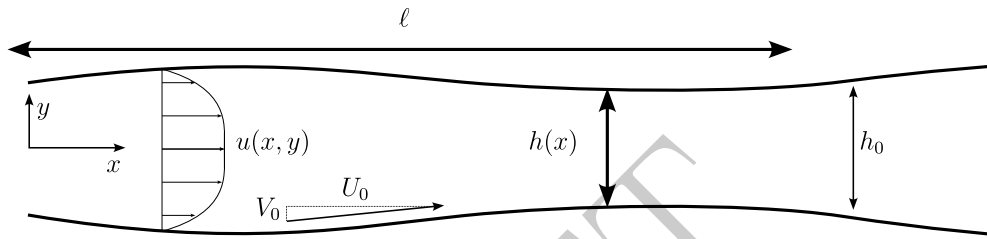


Figure 6.4: Flow in a thin channel/slot with varying height

Let the following length and velocity scales be defined,

$$[x] = \ell, [y] = h_0, [u] = U_0, [v] = V_0. \quad (6.4.1)$$

We analyze the flow in the limit of $\epsilon = \frac{h_0}{\ell} \ll 1$. Note that the only cause of y direction velocity is local deflection of the flow due to angle of the walls. Locally, y direction velocity can be estimated as,

$$\frac{V_0}{U_0} = \tan \theta = \left. \frac{dy}{dx} \right|_{wall}. \quad (6.4.2)$$

The above shows that U_0 and V_0 are related via,

$$\frac{V_0}{U_0} = \frac{h_0}{\ell}. \quad (6.4.3)$$

Note that since we are only commenting about characteristic length scales, we can put the *hard* equality sign. By equating scales, we imply that the quantities are of same order of magnitude. Hence, in a channel of height 1 mm, varying over length 1 cm, the typical values of y direction velocity will be 0.1 times the x direction velocity.

Non-dimensionalizing the continuity equation, we obtain,

$$\frac{U_0}{\ell} \frac{\partial u^*}{\partial x^*} + \frac{V_0}{h_0} \frac{\partial v^*}{\partial y^*} = 0 \implies \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0. \quad (6.4.4)$$

Let the pressure scale be defined as $[p] = p_0$. To identify p_0 , we note that pressure forces balance viscous stresses at the walls in a steady flow. Hence,

$$p_0 h_0 = \frac{\mu U_0}{h_0} \ell \implies p_0 = \frac{\mu U_0 \ell}{h_0^2}. \quad (6.4.5)$$

Non-dimensionalization of the y momentum equation yields,

$$\frac{U_0 V_0}{\ell} u^* \frac{\partial v^*}{\partial x^*} + \frac{V_0^2}{h_0} v^* \frac{\partial v^*}{\partial y^*} = -\frac{p_0}{\rho h_0} \frac{\partial p^*}{\partial y^*} + \frac{\nu V_0}{h_0^2} \left(\left(\frac{h_0}{\ell} \right)^2 \frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}} \right). \quad (6.4.6)$$

Upon simplification, we obtain,

$$\epsilon^2 \left(u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} \right) = -\frac{p_0}{\rho U_0^2} \frac{\partial p^*}{\partial y^*} + \frac{\epsilon}{\text{Re}_h} \left(\frac{\partial^2 v^*}{\partial y^{*2}} + \epsilon^2 \frac{\partial^2 v^*}{\partial x^{*2}} \right). \quad (6.4.7)$$

Noting that $\frac{p_0}{\rho U_0^2} = \frac{1}{\epsilon \text{Re}_h}$, we obtain,

$$\epsilon^3 \text{Re}_h \left(u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} \right) = -\frac{\partial p^*}{\partial y^*} + \epsilon^2 \left(\frac{\partial^2 v^*}{\partial y^{*2}} + \epsilon^2 \frac{\partial^2 v^*}{\partial x^{*2}} \right). \quad (6.4.8)$$

Similarly, the dimensionless x momentum equation yields,

$$\epsilon \text{Re}_h \left(u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} \right) = -\frac{\partial p^*}{\partial x^*} + \left(\frac{\partial^2 u^*}{\partial y^{*2}} + \epsilon^2 \frac{\partial^2 u^*}{\partial x^{*2}} \right). \quad (6.4.9)$$

If, along with the geometric condition $\epsilon \ll 1$, $\epsilon \text{Re}_h \ll 1$ is satisfied, then at the leading order $\mathcal{O}(1)$ we obtain,

$$\frac{\partial p^*}{\partial y^*} = 0, \quad \frac{\partial p^*}{\partial x^*} = \frac{\partial^2 u^*}{\partial y^{*2}}. \quad (6.4.10)$$

Hence, at the leading order, we obtain the governing equations for flow in a thin channel. Converting the equations back to dimensional form and solving, we obtain

$$u = \frac{1}{2\mu} \frac{\partial p}{\partial x} y^2 + C_1(x)y + C_2(x). \quad (6.4.11)$$

Using the no-slip boundary conditions $u = 0$ at $y = \pm \frac{h(x)}{2}$ we obtain,

$$u(x, y) = -\frac{h^2}{8\mu} \frac{dp}{dx} \left(1 - \frac{4y^2}{h^2}\right). \quad (6.4.12)$$

Integrating to obtain the pressure-drop flow-rate relation,

$$\int_{-h/2}^{h/2} u dy = -\frac{h^3}{12\mu} \frac{dp}{dx} = Q. \quad (6.4.13)$$

From this, we obtain the pressure drop between two points x_1 and x_2

$$p_2 - p_1 = -12\mu Q \int_{x_1}^{x_2} \frac{dx}{h^3}. \quad (6.4.14)$$

6.4.1 Reynolds equation

Viscous flow in thin gaps is often encountered in machinery lubrication such as slipper-pad bearings, thrust bearings, ball bearings, gears etc. In Cartesian coordinates, a general case can be represented by figure 6.5. Applying the lubrication approximation $\epsilon \ll 1$ and $\epsilon \text{Re}_h \ll 1$, we obtain the governing equations

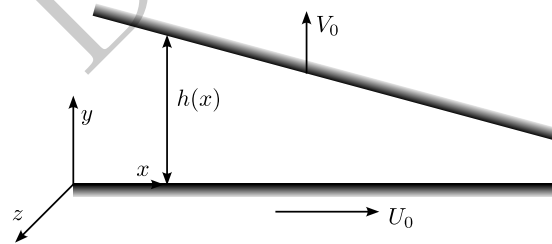


Figure 6.5: A Cartesian coordinate analogue of a lubrication bearing

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}, \quad (6.4.15)$$

$$\frac{\partial p}{\partial z} = \mu \frac{\partial^2 w}{\partial y^2}, \quad (6.4.16)$$

$$\frac{\partial p}{\partial y} = 0. \quad (6.4.17)$$

with boundary conditions $u(x, y = 0) = u(x, y = h(x)) = 0$ and $w(x, y = 0) = w(x, y = h(x)) = 0$. Solving for u and w , we obtain,

$$u = \frac{1}{2\mu} \frac{\partial p}{\partial x} (y^2 - yh) + \left(1 - \frac{y}{h}\right) U_0, \quad w = \frac{1}{2\mu} \frac{\partial p}{\partial z} (y^2 - yh). \quad (6.4.18)$$

To evaluate pressure, we turn to the continuity equation. This highlights an important feature of the incompressible governing equations. Pressure and continuity equation are actually very closely related.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (6.4.19)$$

integrating the above equation in y we obtain,

$$\int_0^h \frac{\partial u}{\partial x} dy + \int_0^h \frac{\partial w}{\partial z} dy = v(0) - v(h) \quad (6.4.20)$$

From boundary conditions, we know that $v(h) = V_0$. Furthermore,

$$\int_0^h \frac{\partial u}{\partial x} dy = \frac{\partial}{\partial x} \int_0^h u dy - u(h) \frac{dh}{dx} = \frac{\partial}{\partial x} \int_0^h u dy, \quad (6.4.21)$$

due to no-slip $u(y = h) = 0$. Substituting for u and w , we further obtain,

$$-V_0 = \frac{U_0}{2} \frac{\partial h}{\partial x} - \frac{1}{12\mu} \left(\frac{\partial}{\partial x} \left(h^3 \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial z} \left(h^3 \frac{\partial p}{\partial z} \right) \right). \quad (6.4.22)$$

After re-arrangement, we obtain the Reynolds equation for lubrication theory which is the governing equation for pressure inside a confined thin gap as considered,

$$\frac{1}{\mu} \left(\frac{\partial}{\partial x} \left(h^3 \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial z} \left(h^3 \frac{\partial p}{\partial z} \right) \right) = 6U_0 \frac{\partial h}{\partial x} + 12 \frac{\partial h}{\partial t}, \quad (6.4.23)$$

where we have used $\frac{\partial h}{\partial t} = V_0$. For known $h(x, t)$, $p(x, z, t)$ can be solved for by solving the above equation at every time instant.

The above equation also describes the viscous adhesion phenomenon that we have all experienced when trying to lift something flat on a wet surface. For $h(x) = h_0$ and V_0 , we obtain,

$$\frac{\partial p}{\partial x} = \frac{12\mu}{h^3} x V_0, \quad (6.4.24)$$

if the gap is open to atmosphere, then we know that $p = p_0$ at $x = \pm \frac{L}{2}$ for some L , which yields,

$$p - p_0 = -\frac{3\mu L^2}{2h^3} \frac{dh}{dt} \left(1 - \frac{4x^2}{L^2}\right) \quad (6.4.25)$$

which shows that $p < p_0$ for $\frac{dh}{dt} > 0$. This implies, the atmospheric pressure p_0 will try to push the flat surface downwards due to lesser pressure inside the fluid filled gap.

DRAFT