

Chapter 7

A short introduction to asymptotics

In the previous chapter we saw a very classic application of the asymptotics to fluid mechanics. Navier-Stokes equations are governing equations for all flows, period. However, often times, we can find *smallness parameters* which can help in simplifying these equations to the point that some analytical solutions can be derived for insights. In this chapter, we will review asymptotics using a few problems, which exemplify the simplification abilities of this class of methods. We will also give a short introduction of singular asymptotic problems. Understanding singular asymptotic problems is a must for a physical understanding of potential flow and boundary layer theory, both being two sides of the same coin of the high-Re flows.

7.1 Domain perturbation problems

In domain perturbation problems, a practical problem can be considered as a fundamental problem with a small change in geometry. For instance, the last topic we covered was lubrication theory, which has applications in tribology (bearings and lubrication). A very typical application is that of a journal bearing which has a shaft inside a housing with lubricating oil/fluid in between. The shaft (or in some cases the housing) rotate relative to each other, resulting in a viscous flow of oil between the two. The pressures can rise to high values with increasing relative rotation speeds, which can increase wear and tear, when ironically, the whole point of using lubricating oil is to reduce wear and tear. In these bearings, the shafts are kept in a eccentric position compared to the housing in a static configuration. As the shaft rotates, there is a pressure force generated due to eccentricity which helps in bearing any normal loads on the shaft.

Figure 7.1 shows a typical configuration which allows study of the problem. The simple analogue of this problem is the Taylor-Couette flow with no eccentricity. The domain perturbation analogue which we treat here is of thin-gap $\epsilon \ll 1, \lambda = \mathcal{O}(1)$. There is a small eccentricity case of the problem also, which may be found in lubrication textbooks (see Leal [1992] for details). The governing equations for steady flow are Navier-Stokes equations in cylindrical coordinates (assuming no flow in z direction)

$$\frac{1}{r} \frac{\partial(r u_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} = 0, \quad (7.1.1)$$

$$u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \left(\frac{\partial u_r}{\partial \theta} - u_\theta \right) = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2} \right), \quad (7.1.2)$$

$$u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \left(\frac{\partial u_\theta}{\partial \theta} + u_r \right) = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \nu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right). \quad (7.1.3)$$

The boundary conditions are

$$u_r = 0, u_\theta = \Omega_1 a \text{ at } r = a, \quad (7.1.4)$$

$$\mathbf{u} \cdot \hat{n} = 0, \mathbf{u} \cdot \hat{t} = \Omega_2 (1 + \epsilon) a \text{ at } r = r_0(\theta). \quad (7.1.5)$$

Due to eccentricity, the outer cylinder surface is not perfectly oriented along the tangential vector, hence the boundary conditions have to be applied to normal and tangential components carefully. The surface can be seen as the surface

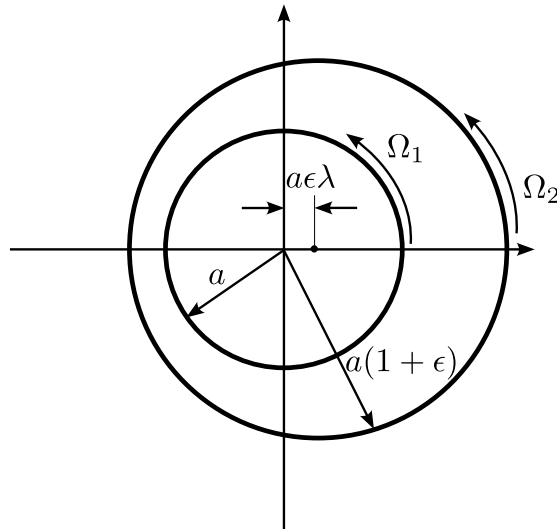


Figure 7.1: Cross-sectional view of a journal bearing problem.

defined by radial distance $r_0(\theta)$ which varies with θ and is given by,

$$r_0 = a\sqrt{(\lambda\epsilon)^2 + (1 + \epsilon)^2 + 2\lambda\epsilon(1 + \epsilon)\cos\theta}. \quad (7.1.6)$$

The normal unit vector can be evaluated as,

$$\hat{n} = \frac{\hat{e}_r - \hat{e}_\theta \frac{1}{r_0} \frac{dr_0}{d\theta}}{\sqrt{1 + \left(\frac{1}{r_0} \frac{dr_0}{d\theta}\right)^2}}. \quad (7.1.7)$$

Similarly, the tangential unit vector becomes,

$$\hat{t} = \frac{\hat{e}_\theta + \hat{e}_r \frac{1}{r_0} \frac{dr_0}{d\theta}}{\sqrt{1 + \left(\frac{1}{r_0} \frac{dr_0}{d\theta}\right)^2}}. \quad (7.1.8)$$

The standard methodology of asymptotics is to find solutions as power series of the smallness parameter. In the current problem, ϵ is the smallness parameter and we intend to find

$$u_\theta = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \dots \quad (7.1.9)$$

$$u_r = v_0 + \epsilon v_1 + \epsilon^2 v_2 + \dots \quad (7.1.10)$$

$$p = p_0 + \epsilon p_1 + \epsilon^2 p_2 + \dots \quad (7.1.11)$$

We consider the following,

$$r_0 = a(1 + \epsilon) + a\epsilon\lambda\cos\theta + \mathcal{O}(\epsilon^2), \quad (7.1.12)$$

$$\hat{n} = \hat{e}_r + \epsilon\lambda\sin\theta\hat{e}_\theta + \mathcal{O}(\epsilon^2), \quad (7.1.13)$$

$$\hat{t} = \hat{e}_\theta - \epsilon\lambda\sin\theta\hat{e}_r + \mathcal{O}(\epsilon^2). \quad (7.1.14)$$

To analyze the equations, we introduce a change of variable from r to $y = r - a$

with the governing equations,

$$\frac{\partial}{\partial y} [(y+a)u_r] + \frac{\partial u_\theta}{\partial \theta} = 0, \quad (7.1.15)$$

$$\rho \left[u_r \frac{\partial u_r}{\partial y} + \frac{u_\theta}{y+a} \left(\frac{\partial u_r}{\partial \theta} - u_\theta \right) \right] = -\frac{\partial p}{\partial y} \quad (7.1.16)$$

$$+ \mu \left[\frac{1}{y+a} \frac{\partial}{\partial y} \left((y+a) \frac{\partial u_r}{\partial y} \right) + \frac{1}{(y+a)^2} \frac{\partial^2 u_r}{\partial \theta^2} - \frac{u_r}{(y+a)^2} - \frac{2}{(y+a)^2} \frac{\partial u_\theta}{\partial \theta} \right], \quad (7.1.17)$$

$$\rho \left[u_r \frac{\partial u_\theta}{\partial y} + \frac{u_\theta}{y+a} \left(\frac{\partial u_\theta}{\partial \theta} + u_r \right) \right] = -\frac{1}{y+a} \frac{\partial p}{\partial \theta} \quad (7.1.18)$$

$$+ \mu \left[\frac{1}{y+a} \frac{\partial}{\partial y} \left((y+a) \frac{\partial u_\theta}{\partial y} \right) + \frac{1}{(y+a)^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{(y+a)^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{(y+a)^2} \right]. \quad (7.1.19)$$

Considering the simpler case of $\Omega_1 = \Omega$ and $\Omega_2 = 0$, we obtain,

$$u_r = 0, \quad u_\theta = \Omega a, \quad \text{at } y = 0, \quad (7.1.20)$$

$$u_r = u_\theta = 0, \quad \text{at } y = h(\theta). \quad (7.1.21)$$

Let us non-dimensionalize the equations to obtain the leading order equations. Let the following scales,

$$[u_\theta] = \Omega a, \quad [y] = \epsilon a, \quad [u_r] = V. \quad (7.1.22)$$

Non-dimensionalizing the continuity equation, we obtain,

$$\frac{1}{\epsilon a} \frac{\partial}{\partial y^*} [(\epsilon a y^* + a) V u_r^*] + a \Omega \frac{\partial u_\theta^*}{\partial \theta} = 0. \quad (7.1.23)$$

We note that the radial direction velocity occurs due to the offset $\epsilon \lambda a$. Hence, we define $V = \epsilon a \Omega$ from which, we obtain,

$$\frac{\partial}{\partial y^*} [(\epsilon y^* + 1) u_r^*] + \frac{\partial u_\theta^*}{\partial \theta} = 0. \quad (7.1.24)$$

Let a pressure scale $[p] = p_0$. We obtain the dimensionless y and θ momentum

equations as,

$$\left(\frac{\rho a^2 \Omega}{\mu}\right) \epsilon \left[\epsilon \left(u_r^* \frac{\partial u_r^*}{\partial y^*} + \frac{u_\theta^*}{1 + \epsilon y^*} \frac{\partial u_r^*}{\partial \theta} \right) - \frac{u_\theta^{*2}}{(1 + \epsilon y^*)} \right] = -\frac{p_0}{\mu \Omega} \frac{\partial p^*}{\partial y^*} + \left[\frac{1}{1 + \epsilon y^*} \frac{\partial}{\partial y^*} \left((1 + \epsilon y^*) \frac{\partial u_r^*}{\partial y^*} \right) + \frac{\epsilon^2}{(1 + \epsilon y^*)^2} \frac{\partial^2 u_r^*}{\partial \theta^2} - \frac{\epsilon^2 u_r^*}{(1 + \epsilon y^*)^2} - \frac{2\epsilon}{(1 + \epsilon y^*)^2} \frac{\partial u_\theta^*}{\partial \theta} \right], \quad (7.1.25)$$

$$\left(\frac{\rho a^2 \Omega}{\mu}\right) \epsilon^2 \left[u_r^* \frac{\partial u_\theta^*}{\partial y^*} + \frac{u_\theta^*}{1 + \epsilon y^*} \frac{\partial u_\theta^*}{\partial \theta} + \epsilon \frac{u_\theta^* u_r^*}{1 + \epsilon y^*} \right] = -\frac{p_0 \epsilon^2}{\mu \Omega} \frac{1}{1 + \epsilon y^*} \frac{\partial p^*}{\partial \theta} + \left[\frac{1}{1 + \epsilon y^*} \frac{\partial}{\partial y^*} \left((1 + \epsilon y^*) \frac{\partial u_\theta^*}{\partial y^*} \right) + \frac{\epsilon^2}{(1 + \epsilon y^*)^2} \left(\frac{\partial^2 u_\theta^*}{\partial \theta^2} - u_\theta^* \right) + \frac{2\epsilon^3}{(1 + \epsilon y^*)^2} \frac{\partial u_r^*}{\partial \theta} \right]. \quad (7.1.26)$$

The dimensionless boundary conditions become,

$$u_r^* = 0, u_\theta^* = 1 \text{ at } y^* = 0 \quad (7.1.27)$$

and

$$u_r^* = u_\theta^* = 0 \text{ at } y^* = h^*(\theta). \quad (7.1.28)$$

Considering leading order equations and dropping $()^*$ notation for simplicity, we note that the curvature in the problem is irrelevant in the thin-gap limit which is defined as,

$$\epsilon \ll 1, \quad \frac{\rho a^2 \Omega}{\mu} \epsilon \ll 1. \quad (7.1.29)$$

The dimensionless governing equations are identical to the lubrication equations derived in the Cartesian coordinates,

$$\frac{\partial u_r}{\partial y} + \frac{\partial u_\theta}{\partial \theta} = 0, \quad (7.1.30)$$

$$\frac{\partial^2 u_\theta}{\partial y^2} - \frac{\partial p}{\partial \theta} = 0, \quad (7.1.31)$$

$$\frac{\partial p}{\partial y} = 0. \quad (7.1.32)$$

Note that we have also taken pressure scale $p_0 = \frac{\mu \Omega}{\epsilon^2}$.

7.2 Regular asymptotic differential equations

In the previous section, we saw that after non-dimensionalization, terms which are revealed as small are not the highest order terms in the differential equation.

Such type of different equations can be solved by so-called regular asymptotic expansions, and hence are often referred as *regular asymptotic equations* or *regular asymptotic problems*. An important feature of regular asymptotic problems is, the solutions obtained by naively dropping out the ϵ terms are *uniformly valid*, i.e., they give the appropriate order approximation of the actual analytical solution valid over the whole domain. For instance, consider the ODE,

$$\frac{d^2u}{dy^2} + \frac{du}{dy} + \epsilon = 0, \quad u(0) = 0, \quad u(1) = 1, \quad \epsilon \ll 1. \quad (7.2.1)$$

The complete analytical solution of the problem is obtained using integrating factor,

$$\frac{du}{dy} + u + \epsilon y = C_1 \implies \frac{d}{dy}(ue^y) = (C_1 - \epsilon y) e^y. \quad (7.2.2)$$

The complete solution is

$$u(y) = C_2 e^{-y} + C_1 - \epsilon(y - 1) \quad (7.2.3)$$

Substituting the boundary conditions, we obtain,

$$u(y) = \frac{e + \epsilon}{e - 1} - \frac{(1 + \epsilon)e^{-y+1}}{e - 1} - \epsilon(y - 1). \quad (7.2.4)$$

In the limit $\epsilon \rightarrow 0$, the above solution becomes,

$$\lim_{\epsilon \rightarrow 0} u(y) = \frac{e(1 - e^{-y})}{e - 1}, \quad (7.2.5)$$

which is exactly the solution obtained by just naively dropping out the ϵ term in Eq. (7.2.1). Hence, the solution obtained by ignoring the ϵ terms in governing equations is indeed an approximation of the actual solution in the whole domain. The error in approximation is of the same order throughout the domain, hence we call the approximation to be *uniformly valid*.

7.3 Singular asymptotic problems

The situation in the above example is not so easy if ϵ multiplies the $\frac{d^2u}{dy^2}$ term. Consider,

$$\epsilon \frac{d^2u}{dy^2} + \frac{du}{dy} + 1 = 0, \quad u(0) = 0, \quad u(1) = 1, \quad \epsilon \ll 1. \quad (7.3.1)$$

The exact solution of the above equation can be obtained using the integration factor as above,

$$u(y) = \frac{2(1 - e^{-y/\epsilon})}{1 - e^{-1/\epsilon}} - y. \quad (7.3.2)$$

The exact solution can not be fully recovered if we naively drop the ϵ term in Eq. (7.3.1). The intuitive reasoning is, after dropping the ϵ term, the equation is first order and can not necessarily satisfy both the boundary conditions. The solution we will get may be an approximation of the exact solution *only within some subset of the domain*. And this is the whole nuance of singular asymptotic problems. As a result, a concept of an *inner solution* and an *outer solution* is introduced.

7.3.1 Outer solution

Naively dropping the ϵ term yields the leading order approximation of the exact solution in the outer region. Note that to conclude that $\epsilon \frac{d^2 u}{dy^2} \ll 1$, we state that $\frac{d^2 u}{dy^2}$ is $\mathcal{O}(1)$. This implies that we are looking at variations of u over large distances. Note that through boundary conditions, change in u has a scale of 1. This automatically suggests that $\epsilon \frac{d^2 u}{dy^2} \ll 1$ can hold far away from $y \rightarrow 0$. Dropping the ϵ term, we obtain,

$$u_o = C - y. \quad (7.3.3)$$

Satisfying $u(1) = 1$ from this, we obtain $u_o = 2 - y$. If we take $\lim_{y \rightarrow 1}$ for the analytical solution, we do obtain $2 - y$. This is called the outer solution of the equation.

7.3.2 Inner solution

The inner solution is valid when y itself is very small and the scale of $\frac{d^2 u}{dy^2}$ can not be claimed as $\mathcal{O}(1)$. Instead, we introduce another *scaled variable*

$$y = \epsilon \xi, \quad (7.3.4)$$

where ξ takes values from 0 to ∞ . As ξ increases to ∞ , we impose that the growth of ξ is ϵ^r where $r < 1$ such that y retains some finite small value. This ξ may be seen as a *stretched variable*, where we are stretching the y coordinates to zoom further inside them, close to 0. Converting the ODE to stretched variable,

we obtain,

$$\frac{d^2u}{d\xi^2} + \frac{du}{d\xi} + \epsilon = 0. \quad (7.3.5)$$

Dropping the ϵ term now, we obtain

$$u_I = C_1 + C_2 e^{-\xi}. \quad (7.3.6)$$

For $\xi \rightarrow 0$, y also approaches 0, hence, $u(\xi = 0) = 0$ is the boundary condition on one side in the stretched variable which yields $u_I = C_1(1 - e^{-\xi})$.

7.3.3 Asymptotic Matching

The other boundary condition for the inner solution u_I is obtained using $y \rightarrow 0$ limit of u_o , the outer solution. As the name suggests, this is the asymptotic matching procedure.

$$\lim_{y \rightarrow 0} u_O = \lim_{\xi \rightarrow \infty} u_I, \quad (7.3.7)$$

which yields the inner solution now,

$$u_I = 2(1 - e^{-\xi}) = 2(1 - e^{-y/\epsilon}). \quad (7.3.8)$$

If we evaluate $\lim_{y \rightarrow 0}$ of the exact solution, we indeed obtain this limit.

7.3.4 Composite solution

We note that the inner and the outer solutions both were obtained by dropping the $\mathcal{O}(\epsilon)$ terms, hence, both yield the approximation to the analytical solution in their respective zones up to $\mathcal{O}(\epsilon)$. This is the only reason why, we can combine them to obtain a *uniformly valid approximation* to the exact solution,

$$u(y) = u_I + u_O - \text{common part} + \mathcal{O}(\epsilon). \quad (7.3.9)$$

The above yields the leading order approximation of the exact solution as,

$$u(y) = 2 - y + 2(1 - e^{-y/\epsilon}) - 2 + \mathcal{O}(\epsilon) = 2(1 - e^{-y/\epsilon}) - y + \mathcal{O}(\epsilon), \quad (7.3.10)$$

which is the leading order approximation of the exact solution, and is often referred as the *composite solution*.

7.3.5 Rotary viscous coupling - application of asymptotic matching

Consider fluid filled in a thin gap of height h in a container with a piston rotating at Ω above it. Assuming only circumferential axisymmetric flow $u_\theta(r, z)$, let us use the following scales for non-dimensionalization,

$$[r] = R, [z] = h, [u_\theta] = R\Omega, \epsilon = \frac{h}{R} \ll 1. \quad (7.3.11)$$

Simplified equation for steady circumferential axisymmetric flow is

$$\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (ru_\theta) \right) + \frac{\partial^2 u_\theta}{\partial z^2} = 0. \quad (7.3.12)$$

The boundary conditions are,

$$u_\theta(r, z = 0) = 0, u_\theta(r, z = h) = r\Omega, u_\theta(r = R, z) = 0. \quad (7.3.13)$$

Non-dimensionalization yields,

$$\epsilon^2 \frac{\partial}{\partial r^*} \left(\frac{1}{r^*} \frac{\partial}{\partial r^*} (r^* u_\theta^*) \right) + \frac{\partial^2 u_\theta^*}{\partial z^{*2}} = 0. \quad (7.3.14)$$

Outer solution

In the limit $\epsilon \rightarrow 0$, we can drop the radial stress term if radial stress term is not very large. Given the nature of the problem, clearly the radial variation will be maximum near the lateral walls of the cavity. Naturally, the obtained solution

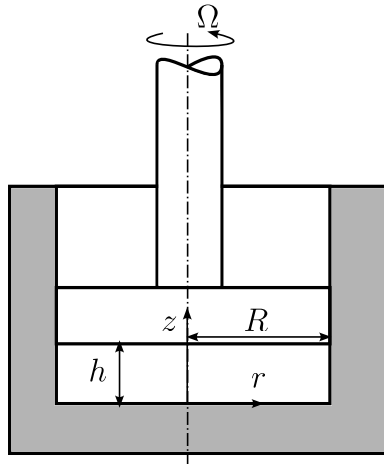


Figure 7.2: Rotary viscous coupling

can then only satisfy the $r = 0$ boundary condition. Dropping out the ϵ term, we obtain,

$$u_\theta^* = A(r^*)z^* + B. \quad (7.3.15)$$

Using the boundary condition at $z^* = 1$, $u_\theta^* = r^*$ and no-slip condition at $z^* = 0$ we obtain the dimensionless solution

$$u_\theta^* = r^* z^*. \quad (7.3.16)$$

The above solution is the *outer solution*. It is valid only when r^* is sufficiently smaller than 1. As r^* approaches 1, the no-slip boundary condition is violated at $r^* = 1$, to satisfy which, we need to solve for the inner solution.

Inner solution

To find the inner solution close to the lateral surface, we choose a coordinate $y = R - r$ and non-dimensionlize this with h instead of R to obtain,

$$y^* = \frac{1 - r^*}{\epsilon}. \quad (7.3.17)$$

Substituting for r^* , we obtain the re-scaled governing equation as,

$$\frac{\partial}{\partial y^*} \left(\frac{1}{1 - \epsilon y^*} \frac{\partial}{\partial y^*} [(1 - \epsilon y^*) u_\theta^*] \right) + \frac{\partial^2 u_\theta^*}{\partial z^{*2}} = 0. \quad (7.3.18)$$

In the lim, $\epsilon \rightarrow 0$, we obtain the following leading order equation,

$$\frac{\partial^2 u_\theta^*}{\partial y^{*2}} + \frac{\partial^2 u_\theta^*}{\partial z^{*2}} = 0. \quad (7.3.19)$$

The boundary conditions to be used are,

$$u_\theta^*(y^*, z^* = 0) = 0, \quad u_\theta^*(y^*, z^* = 1) = 1 + \mathcal{O}(\epsilon), \quad u_\theta^*(y^* = 0, z^*) = 0. \quad (7.3.20)$$

Since the inner equation is a Laplacian equation, another boundary condition is required, which is the matching condition,

$$u_{\theta,I}^*(y^* \rightarrow \infty, z^*) = u_{\theta,O}^*(r^* \rightarrow 1, z^*) = z^*. \quad (7.3.21)$$

Using separation of variables, the inner solution can be obtained, which can be further combined with the outer solution to obtain a uniformly valid approximation [Panton, 2024].

7.4 High Re flows as singular asymptotic problems

From the dimensionless equation,

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* = -\nabla^* p^* + \frac{1}{\text{Re}} \nabla^{*2} \mathbf{u}^*, \quad (7.4.1)$$

we note that high Re flows are governed by a singular asymptotic problem. The small parameter $\frac{1}{\text{Re}}$ multiplies the highest order term. Hence, naively dropping out the ϵ term for approximating the solution only gives a part of the solution. The equation obtained is,

$$\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* = -\nabla^* p^* \quad (7.4.2)$$

at leading order. The above equation is called the Euler's equation and together with the continuity equation $\nabla^* \cdot \mathbf{u}^* = 0$, it governs an inviscid flow. However, all high Re flows are not governed by inviscid flow equations. Very close the boundary of any hard-wall surface, to satisfy the no-slip condition, large gradients in velocity occur. These gradients mandate rescaling of the equation to obtain the so-called *boundary layer* equations as an inner regime equation. Figure 7.3 summarizes these outer and inner regimes. These two regimes are treated separately in detail in the two chapters below.

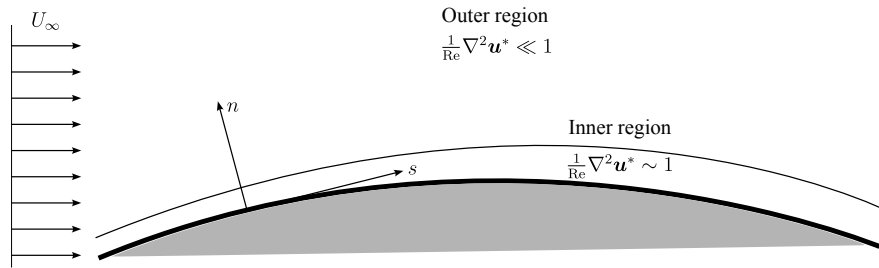


Figure 7.3: Inviscid flow (outer region) and boundary layer (inner region), separated by characteristic length scales.

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