

Chapter 8

High Re flows (inviscid/outer solution)

If the characteristic length scales in all directions are taken same, and the corresponding Reynolds number is large, then the leading order governing equations in the dimensionless form are

$$\nabla^* \cdot \mathbf{u}^* = 0, \frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* = -\nabla^* p^*, \quad (8.0.1)$$

or in the dimensional form,

$$\nabla \cdot \mathbf{u} = 0, \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla p}{\rho}. \quad (8.0.2)$$

As discussed in the previous chapter, these are outer regime or far-field equations. Very close to the boundary, we have to be careful (discussed in detail in the next chapter). Equations 8.0.2 govern inviscid flow of a fluid. The momentum equation is a first order PDE now. In terms of boundary conditions, no-slip boundary conditions can not be sustained by the flow anymore and they are neglected while solving for inviscid flow past a body.

8.1 Steady inviscid flow

For steady incompressible inviscid flow, we note that the governing equations can be written as,

$$\nabla \left(\frac{u^2}{2} + \frac{p}{\rho} \right) = \mathbf{u} \times \boldsymbol{\omega}, \quad \nabla \cdot \mathbf{u} = 0. \quad (8.1.1)$$

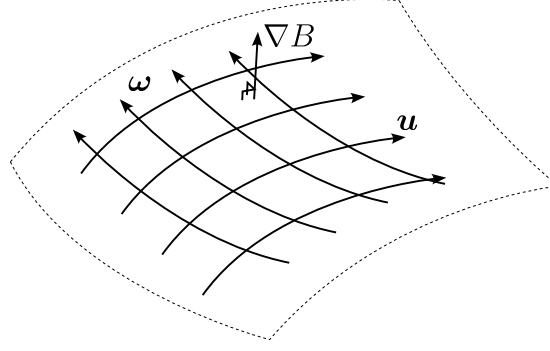


Figure 8.1: A surface spanned by vortex lines and streamlines in a steady inviscid flow

If we define a variable $B = \frac{u^2}{2} + \frac{p}{\rho}$, then we note that the normal to the variable is perpendicular to the lines parallel to $\mathbf{u}$ as well as $\boldsymbol{\omega}$. These two family of lines are streamlines and vortex lines, respectively. Hence, on the surface of these two lines, B is constant (or the surface is the iso-surface of B). This yields the steady state Bernoulli's equation, which is valid only along streamlines or vortex lines in an inviscid flow (see figure 8.1).

8.2 Irrotational flows/Ideal flows

For unsteady flows, a general form Bernoulli's equation can be derived in a very specific case of inviscid flows, namely the ideal flows. Ideal flows are flows in which vorticity is 0. Such flows (like most flows) are approximations (sometimes crude) of the real world flow physics. Yet, they help us reach some well-established practically relevant results. Ideal flows can be justified if we consider the following result. Consider circulation Γ along a closed loop of material coordinates. As the fluid flows, we maintain the closed loop by joining the same material coordinates. Invoking the notations used in chapter two,

$$\Gamma = \oint_{c(\mathbf{x})} \mathbf{u} \cdot d\mathbf{x}. \quad (8.2.1)$$

Now, let consider the time derivative of Γ

$$\frac{d\Gamma}{dt} = \frac{d}{dt} \oint_{c(\mathbf{x})} \mathbf{u} \cdot d\mathbf{x} = \frac{d}{dt} \oint_{C(\mathbf{X})} \mathbf{u} \cdot \mathbf{F} d\mathbf{X} = \oint_{C(\mathbf{X})} \left(\frac{Du_i}{Dt} F_{iJ} + u_i \frac{dF_{iJ}}{dt} \right) dX_J. \quad (8.2.2)$$

The second term can be simplified as,

$$\oint_{C(\mathbf{X})} u_i \frac{dF_{iJ}}{dt} dX_J = \oint_{C(\mathbf{X})} u_i \frac{\partial u_i}{\partial x_p} F_{pJ} dX_J = \oint_{c(\mathbf{x})} \frac{\partial}{\partial x_p} \left(\frac{u_i u_i}{2} \right) dx_p = 0. \quad (8.2.3)$$

The first term can be simplified as,

$$\oint_{C(\mathbf{X})} \frac{Du_i}{Dt} F_{iJ} dX_J = -\frac{1}{\rho} \oint_{c(\mathbf{x})} \frac{\partial p}{\partial x_i} dx_i = 0. \quad (8.2.4)$$

The above calculation shows that rate of change of circulation Γ around a material curve is 0 in an inviscid flow. For a general viscous flow, the rate of change of circulation is

$$\frac{d\Gamma}{dt} = \oint_{c(\mathbf{x})} \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} dx_i = \oint_{c(\mathbf{x})} \nu (\nabla^2 \mathbf{u}) \cdot d\mathbf{x}, \quad (8.2.5)$$

which is the full statement of the Kelvin's theorem. For inviscid flows $\frac{d\Gamma}{dt} = 0$. Constant Γ highlights that if there is no vorticity initially within a closed loop formed by material coordinates, then an inviscid flow does not develop any vorticity. Figure 8.2 highlights this. A uniform flow past a body has no vorticity far-upstream. Hence, in the outer region of the flow where inviscid equations are valid at the leading order, there is no vorticity in the flow. On the other hand, if there is vorticity in the flow far-upstream, then the outer region of the flow past a body can not be treated as an irrotational flow. It is still largely inviscid in nature if the vorticity far-upstream is not very strong. Note that if vorticity is strong and fluctuating upstream, then the flow can not be modeled as inviscid anywhere since $\frac{1}{\text{Re}} \nabla^{*2} \mathbf{u}^* = -\frac{1}{\text{Re}} \nabla^* \times \boldsymbol{\omega}^*$ may not be negligible even in the so-called outer region. If ideal flow can be assumed, then we note that $\nabla \times \mathbf{u} = 0$ everywhere.

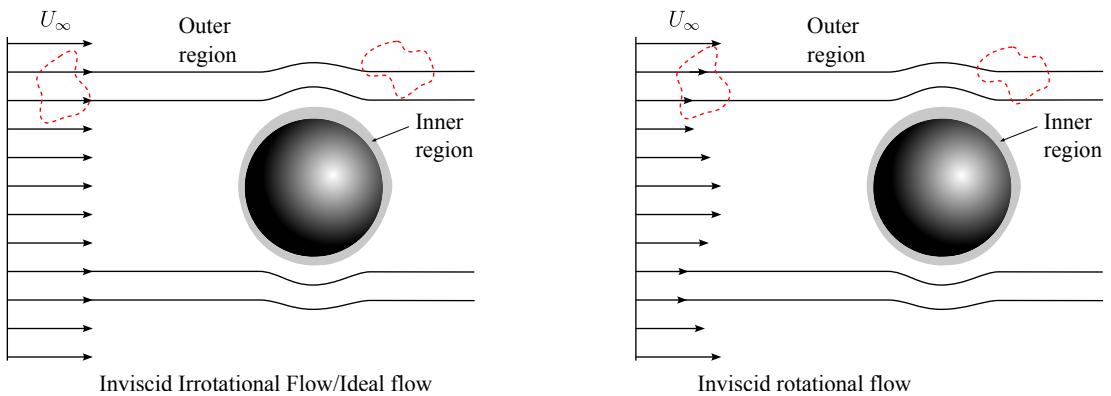


Figure 8.2: Irrotational vs rotational inviscid flows

Hence, we can write velocity as

$$\mathbf{u} = \nabla\phi, \quad (8.2.6)$$

where ϕ is called the scalar velocity potential or just velocity potential. Substituting this in the inviscid flow governing equations, we obtain,

$$\nabla^2\phi = 0, \quad \nabla \left(\frac{\partial\phi}{\partial t} + \frac{|\nabla\phi|^2}{2} + \frac{p}{\rho} \right) = 0, \quad (8.2.7)$$

which yields,

$$\frac{\partial\phi}{\partial t} + \frac{|\nabla\phi|^2}{2} + \frac{p}{\rho} = C(t), \quad (8.2.8)$$

combined with the Laplacian equation $\nabla^2\phi = 0$. Equation (8.2.8) is the unsteady Bernoulli's equation. It is applicable everywhere INSIDE AN IRROTATIONAL FLOW (not only restricted to streamlines or vortex lines). The usual procedure to solve for irrotational flows is to solve $\nabla^2\phi = 0$ and then use the Bernoulli's equation to solve for pressure. Pressure on the surface of an object can then be integrated to compute the force due to inviscid flow.

8.3 Irrotational flows in 2D

For solving 2D ideal flows, complex variables can be used. The velocity potential ϕ satisfies the Laplacian. From complex calculus, real and imaginary parts of any complex analytic function satisfy the Laplacian equation. For 2D ideal incompressible flows, the streamfunction also satisfies the Laplacian equation. Hence, we can form a complex potential $F(z) = \phi + i\psi$. We define a complex velocity

$$w(z) = \frac{dF}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} - i\frac{\partial\phi}{\partial y} = u - iv = |u|e^{i\alpha}. \quad (8.3.1)$$

Even though counter-intuitive, complex functions simplify the algebra considerably. For instance, a uniform stream with speed U and magnitude α has the complex potential

$$F(z) = Uze^{-i\alpha} \implies W = U \cos \alpha - iU \sin \alpha = u - iv. \quad (8.3.2)$$

We can also compute velocities in polar coordinates. Using

$$u = u_r \cos \theta - u_\theta \sin \theta, \quad v = u_r \sin \theta + u_\theta \cos \theta, \quad (8.3.3)$$

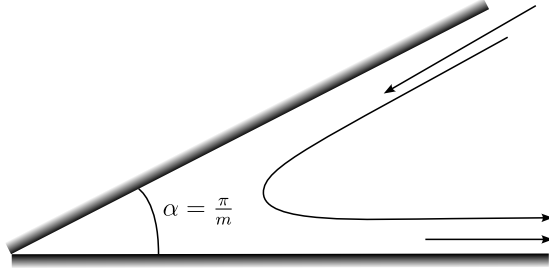


Figure 8.3: Inviscid flow in a wedge of angle $\frac{\pi}{m}$.

we find

$$W = (u_r(r, \theta) - iu_\theta(r, \theta)) e^{-i\theta}. \quad (8.3.4)$$

8.3.1 Ideal flow in a wedge

Now consider a straightforward extension of complex potential for uniform flows. Let

$$F(z) = Az^n. \quad (8.3.5)$$

Complex velocity is given by $W = Anz^{n-1}$. Using polar coordinates $z = re^{i\theta}$,

$$W = nAr^{n-1}(\cos(n\theta) + i\sin(n\theta)) e^{-i\theta}, \quad (8.3.6)$$

which yields $u_r = nAr^{n-1} \cos(n\theta)$ and $u_\theta = -nAr^{n-1} \sin(n\theta)$. Note that $u_\theta = 0$ at $\theta = 0, \frac{\pi}{n}$. We can plot the streamlines to confirm that this is indeed potential flow in a wedge of angle $\frac{\pi}{n}$. Let us compute this flow in a wedge of angle $\frac{\pi}{n}$ by solving the Laplacian. The governing equation in 2D polar coordinates is

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0. \quad (8.3.7)$$

Separating the variables $\phi = R(r)T(\theta)$, we obtain,

$$\frac{r^2 R'' + rR'}{R} + \frac{T''}{T} = 0 \implies \frac{T''}{T} = -n^2 \implies T = A_n e^{in\theta} + B_n e^{-in\theta}. \quad (8.3.8)$$

We do not specify what n is at this moment. However, in polar coordinate problems we always require $T(\theta + 2\pi) = T(\theta)$, i.e., the solution should be periodic in θ . Hence, $n = 0, 1, 2, 3, \dots$. This yields the following solutions for $R(r)$,

$$R = C_n r^n + D_n r^{-n} \text{ for } n > 0 \text{ and } R = C_0 + D_0 \ln r \text{ for } n = 0. \quad (8.3.9)$$

The general solution is

$$\phi = C_0 + D_0 \ln r + \sum_{n=1}^{\infty} (A_n e^{in\theta} + B_n e^{-in\theta}) (C_n r^n + D_n r^{-n}). \quad (8.3.10)$$

For $u_r = \frac{\partial \phi}{\partial r}$ to be finite at $r \rightarrow 0$, we obtain $D_0 = D_n = 0$. For $u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = 0$ at $\theta = 0$, we obtain,

$$\sum_{n=1}^{\infty} in C_n (A_n e^{in\theta} - B_n e^{-in\theta}) r^n = 0 \text{ at } \theta = 0. \quad (8.3.11)$$

The only non-trivial solution possible is if $A_n = B_n$ in the above equation. Hence,

$$\phi = C_0 + \sum_{n=1}^{\infty} A_n r^n \cos(n\theta), \quad (8.3.12)$$

where we have subsumed all constants in A_n . Imposing boundary condition $u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = 0$ at $\theta = \pi/m$, we obtain,

$$\sin\left(\frac{n\pi}{m}\right) = 0. \quad (8.3.13)$$

This yields the eigenvalues $n = m, 2m, 3m, \dots$. Hence the general solution becomes

$$\phi = C_0 + \sum_{k=1}^{\infty} A_k r^{km} \cos(km\theta). \quad (8.3.14)$$

Figure 8.4 shows the visualization of what each of these k solution independently represents. For a general case, some u_r or u_θ has to be specified at some radial location. Let us say $u_r = f(\theta)$ at $r = r_0$. Then

$$\sum_{k=1}^{\infty} (km) A_k r_0^{km-1} \cos(km\theta) = f(\theta), \quad (8.3.15)$$

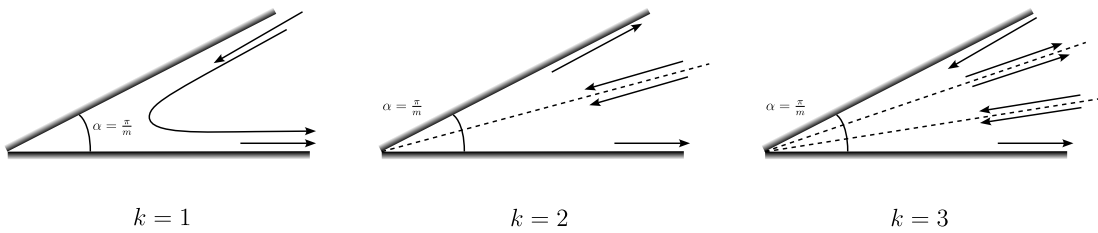


Figure 8.4: Physical visualization of each of k solution in Eq. 8.3.14

from which A_k can be solved for by using cosine series expansion of $f(\theta)$. For a single turn inviscid flow in a wedge, the solution is simply

$$\phi = C_0 A r^m \cos(m\theta). \quad (8.3.16)$$

It is straightforward to confirm that the corresponding complex potential is simply Az^m for this ϕ . This should show how powerful complex potential method can be to compute velocities in ideal flows. It is not always straightforward to visualize the physical picture of the inviscid flow represented by a particular complex potential though. If this physical picture can be realized, then the complex potential method is powerful. The reader is encouraged to look at various textbooks for solving basic planar potential flows using complex variables (see [Panton \[2024\]](#), [Milne-Thomson \[1996\]](#), [Currie and Currie \[2002\]](#)). The power of complex variables is exemplified in transforming flow past cylinders to flow past various airfoils, which is applied in aerodynamics industry to this date!

8.3.2 Ideal flow past a wavy wall

As clear from motivation, ideal flows only constitute a part of the solution (to the problem of high Reynolds number flow past a body). Usually, for symmetric shapes (as is clear from ideal flow past cylinder example), the ideal flow does not yield any drag. This can be further exemplified by considering ideal flow past a wavy wall as shown in figure 8.5. The governing equation remains $\nabla^2\phi = 0$. The boundary condition satisfied by the ideal flow is $\mathbf{u} \cdot \hat{n} = 0$ at the wall, i.e.,

$$\mathbf{u} \cdot \hat{n} = 0 \text{ at } y = \epsilon \ell \sin(\alpha x). \quad (8.3.17)$$

Let the perturbations in the otherwise uniform flow due to wavy wall be denoted by $()'$. Then we can decompose the velocity potential as,

$$\phi = U_\infty x + \epsilon \phi', \quad (8.3.18)$$

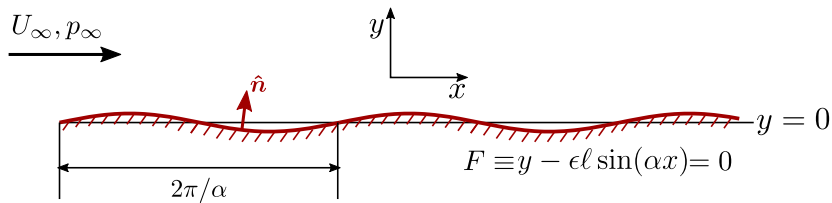


Figure 8.5: Ideal flow of speed U_∞ past a wavy wall.

where $\nabla^2 \phi' = 0$. Given the wall is infinitely long and wavy, we consider,

$$\phi'(x, y) = \hat{\phi}(y)e^{ikx}, \quad (8.3.19)$$

which yields,

$$\frac{d^2 \hat{\phi}}{dy^2} - k^2 \hat{\phi} = 0 \implies \hat{\phi} = Ae^{ky} + Be^{-ky}. \quad (8.3.20)$$

For bounded solution at $y \rightarrow \infty$, we obtain,

$$\phi' = Be^{-ky}e^{ikx}. \quad (8.3.21)$$

Technically, ϕ' is real and a common way of writing it is,

$$\phi' = Ce^{-ky}e^{ikx} + C^*e^{-ky}e^{-ikx}, \quad (8.3.22)$$

where C^* denotes complex conjugate of C . The constant C needs to be obtained. Turning to boundary condition, we obtain,

$$-\left(U_\infty + \epsilon \frac{\partial \phi'}{\partial x}\right) \alpha \ell \cos(\alpha x) + \epsilon \frac{\partial \phi'}{\partial y} = 0 \text{ at } y = \epsilon \sin(\alpha x), \quad (8.3.23)$$

which yields,

$$\epsilon U_\infty \alpha \ell \cos(\alpha x) + \mathcal{O}(\epsilon^2) = \epsilon \frac{\partial \phi'}{\partial y}. \quad (8.3.24)$$

Taylor expanding the terms, we can apply the above equation at $y = 0$ within linear approximation, hence, upon dropping $\mathcal{O}(\epsilon^2)$ terms, we obtain

$$U_\infty \alpha \ell \cos(\alpha x) = \frac{\partial \phi'}{\partial y} \text{ at } y = 0. \quad (8.3.25)$$

Using the solution of ϕ' , we obtain,

$$U_\infty \alpha \ell \cos(\alpha x) = -k \left(Ce^{ikx} + C^*e^{-ikx} \right), \quad (8.3.26)$$

which yields $C = C^* = -\frac{U_\infty \ell}{2}$ and $k = \alpha$. Thus ϕ' is,

$$\phi' = -U_\infty \ell \cos(\alpha x)e^{-\alpha y}, \quad (8.3.27)$$

which yields the leading order velocity due to wavy wall as,

$$u = U_\infty + \epsilon U_\infty \alpha \sin(\alpha x)e^{-\alpha y} + \mathcal{O}(\epsilon^2), \quad v = \epsilon U_\infty \alpha \cos(\alpha x)e^{-\alpha y} + \mathcal{O}(\epsilon^2). \quad (8.3.28)$$

By writing $+\mathcal{O}(\epsilon^2)$, it is always good to remind oneself that the solution obtained is only correct till leading order. There may be further more complicated corrections which we are not interested in so far. To compute drag, we evaluate the pressure at the wavy wall using the Bernoulli's equation,

$$\frac{p}{\rho} + \frac{|\mathbf{u}|^2}{2} = \frac{p_\infty}{\rho} + \frac{U_\infty^2}{2}, \quad (8.3.29)$$

which yields

$$p = p_\infty + \epsilon \rho U_\infty^2 \alpha \ell \sin(\alpha x) + \mathcal{O}(\epsilon^2), \quad (8.3.30)$$

at the wall $y = 0$. We note that the integration of $-p\hat{n}$ yields 0 at leading order, highlighting that there is no drag on the wavy wall due to inviscid flow. In reality, we obviously expect drag on even a flat wall. This drag can be obtained only once we consider the boundary layers (the inner solution).

8.3.3 Orthogonality of potential contours

In 2D ideal flows, the contour lines of ϕ and the stream function ψ are orthogonal. This can be shown as follows. Along a contour of $\phi = \text{const.}$,

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = 0 \implies \frac{dy}{dx} = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y} = -\frac{u}{v}. \quad (8.3.31)$$

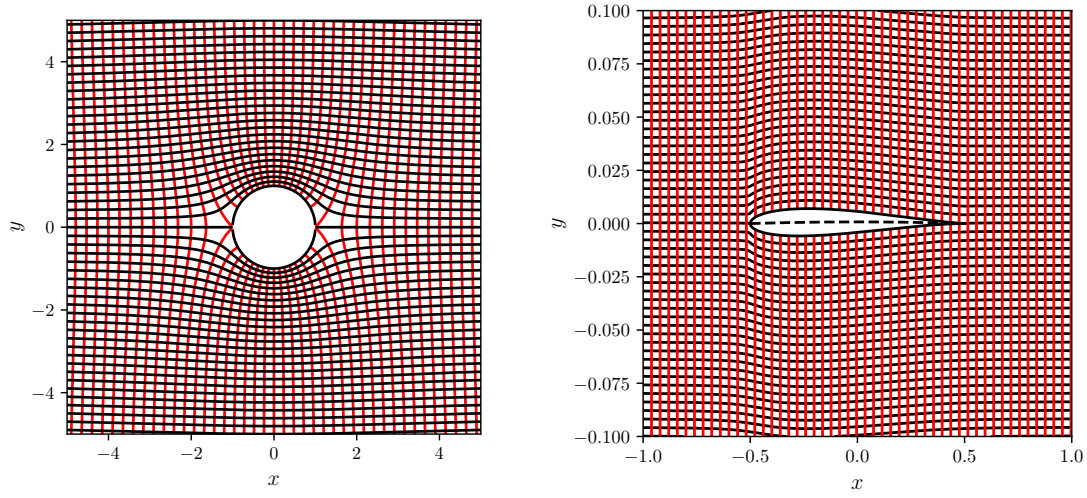
Similarly, along a contour of $\psi = \text{const.}$,

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = 0 \implies \frac{dy}{dx} = -\frac{\partial\psi/\partial x}{\partial\psi/\partial y} = \frac{v}{u}. \quad (8.3.32)$$

Since the product of slopes is always -1 , the contours intersect at 90° . For an ideal flow of speed U_0 past a cylinder of radius r_0 , the complex potential is given by

$$F(z) = U_0 z + \frac{U_0 r_0^2}{z}, \quad (8.3.33)$$

where $z = x + iy$. Figure 8.6b shows the contours of ψ and ϕ for flow past a cylinder and Joukowski airfoil (with values $r_0 = 1$ and $U_0 = 1$).



(a) Iso-contours of ψ (black) and ϕ (red) for an ideal flow past a cylinder.

(b) Iso-contours of ψ (black) and ϕ (red) for an ideal flow past a Joukowski airfoil ($\epsilon = 0.01, \delta = \frac{3\pi}{4}, c = 0.25$).

8.3.4 Joukowski airfoils

Joukowski airfoils constitute a special class of airfoils in which a circle is mapped on an airfoil type shape. The transformation is given by

$$z = \zeta + \frac{c^2}{\zeta}. \quad (8.3.34)$$

In such a case, a circle with center slightly offset from origin is mapped on an airfoil like shape. Figure 8.7 shows the Joukowski transform. For $\epsilon \ll 1$, an approximate equation of the airfoil can be derived. Let the circle be defined by $\zeta = be^{i\gamma}$. From geometry, we see,

$$b^2 + (\epsilon c)^2 - 2\epsilon cb \cos(\delta - \gamma) = c^2 + (\epsilon c)^2 - 2\epsilon c^2 \cos \delta, \quad (8.3.35)$$

which yields,

$$b^2 - 2\epsilon bc \cos(\delta - \gamma) + 2\epsilon c^2 \cos \delta - c^2 = 0 \quad (8.3.36)$$

From the above, we obtain

$$b = c \left(1 + \epsilon(\cos(\delta - \gamma) - \cos(\delta)) + \mathcal{O}(\epsilon^2) \right). \quad (8.3.37)$$

Using this, we can solve for approximate x, y of the airfoil shape as,

$$x = 2c \cos(\gamma), \quad y = 2\epsilon c \sin(\gamma)(\cos(\delta - \gamma) - \cos(\delta)) \quad (8.3.38)$$

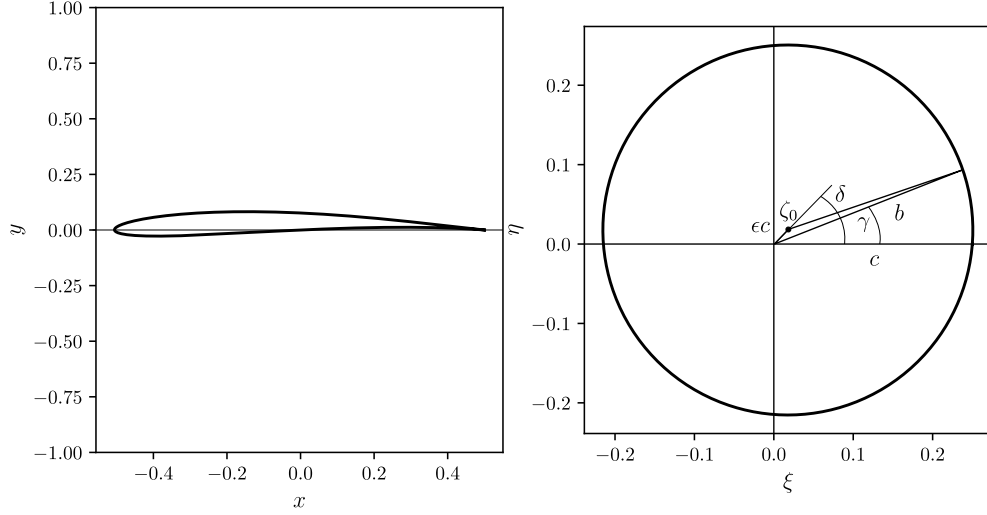


Figure 8.7: Illustration of Joukowski transform which maps a cylinder to an airfoil-like shape.

The flow past this airfoil can be solved by replacing $\zeta - \zeta_0$ in the uniform flow past cylinder potential (which itself is a combination of a doublet and uniform flow),

$$F(z) = U_0(\zeta(z) - \zeta_0) + \frac{U_0 r_0^2}{\zeta(z) - \zeta_0}. \quad (8.3.39)$$

Once the flow is solved, pressure distribution can also be evaluated. Using this pressure distribution, total lift can be obtained (see [Panton \[2024\]](#), [Milne-Thomson \[1996\]](#)).

8.4 Irrotational flows in 3D

For axisymmetric irrotational flows in 3D, we can solve the streamfunction and velocity potential for a few cases. For instance, a point source/sink can be solved assuming purely radial flow. Hence $u_\theta = 0$ if we consider spherical coordinates. Using,

$$u_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}, \quad (8.4.1)$$

we get,

$$\frac{\partial \psi}{\partial r} = 0. \quad (8.4.2)$$

Also, from axisymmetry, we know $u_r = u_r(r)$ only. Hence,

$$\frac{\partial u_r}{\partial \theta} = \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{d\psi}{dr} \right) = 0 \implies \psi = -C_2 \cos \theta + C_1. \quad (8.4.3)$$

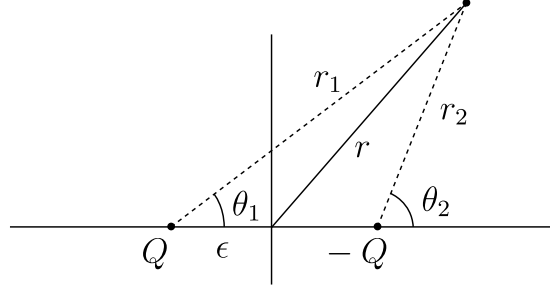


Figure 8.8: 3D doublet

Let $\theta = 0$ be the reference axis, where we set $\psi = 0 \implies C_1 = C_2$. To ensure flow rate Q ,

$$Q = 2\pi \int_0^\pi u_r r^2 \sin \theta d\theta = 2\pi \int_0^\pi \frac{d\psi}{d\theta} d\theta = 2\pi (\psi(\pi) - \psi(0)) = 4\pi C_1, \quad (8.4.4)$$

yielding

$$\psi = \frac{Q}{4\pi} (1 - \cos \theta) \quad (8.4.5)$$

For uniform flow of speed U_0 , we can write in spherical coordinates $\mathbf{u} = U_0 \cos \theta \hat{e}_r - U_0 \sin \theta \hat{e}_\theta$ which yields,

$$\psi = \frac{U_0 r^2 \sin^2 \theta}{2}. \quad (8.4.6)$$

Consider a doublet, which is similar to the usual 2D case, but in 3D. A point source at $x = -\epsilon$ and a point sink of the same flow rate (strength) at $x = \epsilon$. The superposed ψ is given by,

$$\psi = \frac{Q}{4\pi} (\cos \theta_2 - \cos \theta_1) = \frac{Q}{4\pi} \left(\frac{x - \epsilon}{r_2} - \frac{x + \epsilon}{r_1} \right) = \frac{Q}{4\pi r_1 r_2} (x(r_1 - r_2) - \epsilon(r_1 + r_2)). \quad (8.4.7)$$

Using law of sines and geometry, it can be shown that,

$$Q(r_1 - r_2) = \frac{2\epsilon Q \cos \left(\frac{\theta_1 + \theta_2}{2} \right)}{\cos \left(\frac{\theta_2 - \theta_1}{2} \right)}. \quad (8.4.8)$$

In the limit $\epsilon \rightarrow 0$ and $Q\epsilon = \mu$, we obtain the doublet streamfunction as,

$$\psi = \frac{\mu x \cos \theta}{4\pi r^2} - \frac{\mu}{4\pi r} = -\frac{\mu}{4\pi r} \sin^2 \theta. \quad (8.4.9)$$

Similar to 2D, ideal uniform flow past a 3D sphere is modeled using the super-

position of uniform flow stream function and the doublet streamfunction,

$$\psi = \frac{U_0 r^2 \sin^2 \theta}{2} - \frac{\mu}{4\pi r} \sin^2 \theta + C, \quad (8.4.10)$$

where C is some arbitrary constant. The radial velocity is given by

$$u_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} = U_0 \cos \theta - \frac{\mu}{2\pi r^3} \cos \theta. \quad (8.4.11)$$

At $\theta = \pi$, the flow is purely radial and hits a stagnation point on the sphere where $u_r = 0$. The radius of the sphere is hence related to the strength of the doublet and free stream speed as,

$$r_0 = \left(\frac{\mu}{2\pi U_0} \right)^{1/3} \implies \mu = 2\pi U_0 r_0^3. \quad (8.4.12)$$

Hence, ψ becomes,

$$\psi = \frac{1}{2} U_0 r_0^2 \sin^2 \theta \left(\left(\frac{r}{r_0} \right)^2 - \left(\frac{r}{r_0} \right)^{-1} \right) + C. \quad (8.4.13)$$

We take $C = 0$ so that $\psi = 0$ at $r = r_0$. It is instructive to compare this with the Stokes stream function found for very small Reynolds number flows past a sphere.

Ideal flow theory can also be used to describe flow due to water waves. The surface of the water is exposed to atmosphere (constant pressure). Slight perturbations in the water surface result in flow patterns underneath. To leading order, these flow patterns can be explained by ideal flow theory, i.e., ignoring the viscous effects. In the chapter below, we finally look at the inner solution and the so-called boundary layer theory.

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