

Chapter 9

High Re flows (boundary layers/inner solution)

In Section 7.3, we saw that after rescaling the y coordinate to $\epsilon\xi$, the second order derivative term in the ODE was retained. The solution obtained was used to satisfy the no-slip condition at $y = 0$ and then asymptotically matched to the outer solution. To solve this inner solution, we need to zoom in close to the solid surfaces. Close to these solid surfaces, a thin layer called the Boundary Layer develops in which viscous effects are not negligible. Figure 9.1 illustrates this boundary layer close to a flat wall as the flow grazes past it.

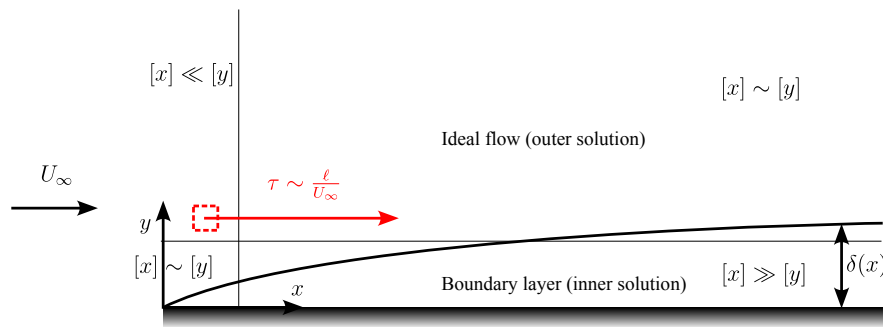


Figure 9.1: Schematic illustrating different regimes of flow which can be separated based on disparity in length scales in two directions x and y .

9.1 Rescaling of normal coordinate :- Laminar boundary layer equations

The ideal flow theory was derived with the asymptotic assumption that $Re_\ell \gg 1$. In boundary layer theory, we distinguish the Reynolds number based on stream-wise coordinate and the wall normal coordinate. This is due to the fact that the characteristic scales in these two directions are very different. Considering figure 9.1, we denote

$$[x] = \ell, [y] = \delta. \quad (9.1.1)$$

This new length scale for wall normal coordinate δ can be thought of as an approximate boundary layer thickness. Note that it is a scale and hence should not be confused with boundary layer thickness function $\delta(x)$ in figure 9.1. Using this length scale, and velocity scales

$$[u] = U_\infty, [v] = V, \quad (9.1.2)$$

we obtain the dimensionless continuity equation as

$$\frac{\partial u^*}{\partial x^*} + \frac{V\ell}{U_\infty\delta} \frac{\partial v^*}{\partial y^*} = 0. \quad (9.1.3)$$

Similar to the lubrication approximation, we note that the y direction velocity scale $V \sim U_\infty \frac{\delta}{\ell}$, i.e., any y direction velocity is due to changing boundary layer thickness downstream. This yields the dimensionless continuity equation as,

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0. \quad (9.1.4)$$

Given we are interested in “zooming-in” closer to the solid surface, it is natural that the boundary layer approximation is $\frac{\delta}{\ell} \ll 1$. This inequality can be derived, as we discuss below after non-dimensionalization of the momentum equations. Before we non-dimensionalize the momentum equations, we need to choose a pressure scale. Now, within lubrication approximation, we argued that pressure changes are due to viscous stresses only, given the inertia is negligible. While, at the outset, the boundary layer approximation has one major feature of the lubrication approximation, i.e. $\frac{\delta}{\ell} \ll 1$. However, inertia is not negligible in boundary layers. The primary difference between boundary layers and lubrication approximation is of the driving mechanism of the flow. In lubrication problems, flows within thin gaps are driven by pressure gradient or moving boundaries. In

boundary layers, which are also thin, flow is driven by its own inertial in the free-stream. Hence, a suitable scale for pressure is $[p] = \rho U_\infty^2$ in the boundary layers. Non-dimensionalizing the y momentum equation, assuming steady flow and $\epsilon = \frac{\delta}{\ell}$ we obtain,

$$\epsilon^2 \left(v^* \frac{\partial v^*}{\partial y^*} + u^* \frac{\partial v^*}{\partial x^*} \right) = -\frac{\partial p^*}{\partial y^*} + \frac{1}{\text{Re}_\ell} \left(\frac{\partial^2 v^*}{\partial y^{*2}} + \epsilon^2 \frac{\partial^2 v^*}{\partial x^{*2}} \right). \quad (9.1.5)$$

Further, non-dimensionalization of x momentum equation yields,

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{\epsilon^2 \text{Re}_\ell} \left(\frac{\partial^2 u^*}{\partial y^{*2}} + \epsilon^2 \frac{\partial^2 u^*}{\partial x^{*2}} \right). \quad (9.1.6)$$

At this moment, a physical relation between δ and ℓ will help us in establishing that indeed $\frac{\delta}{\ell} \ll 1$. Note that our discussion pertains to high Reynolds number flows, so $\text{Re}_\ell \gg 1$ applies. As we think about the boundary layer development, it is helpful to think about a fluid element which is traveling with the free stream velocity U_∞ . It travels the distance equal to $[x] = \ell$ in time $\tau = \frac{\ell}{U_\infty}$. The no-slip condition at the wall results in a diffusion of the *momentum deficit* at the wall upwards. This diffusion is not countered by any geometrical effects for flat plate boundary layers, hence, it can be compared as free diffusion from a boundary. The length till which this momentum deficit diffuses in time τ is $\delta = \sqrt{\nu \tau} = \sqrt{\frac{\nu \ell}{U_\infty}}$, which yields the relation,

$$\frac{\delta}{\ell} = \frac{1}{\sqrt{\text{Re}_\ell}}. \quad (9.1.7)$$

It is interesting to note that $\delta \propto \sqrt{\ell}$. This will be brought out more clearly in Blasius self-similarity solution for flat plates with and without zero pressure gradient. In other analysis (for example jets) where boundary layer approximation is still applicable, we notice that $\delta \propto \ell^\beta$ where β is usually decided by the conditions outside the boundary layer. For instance, in jets, the boundary layer growth is augmented by entrainment of surrounding fluid, which results in $\beta > 1/2$.

Equation (9.1.7) is the boundary layer approximation. Substituting this, we obtain the following boundary layer equations, which are the leading order equations obtained from Eqs (9.1.5) and (9.1.6),

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\partial^2 u^*}{\partial y^{*2}}, \quad (9.1.8)$$

$$\frac{\partial p^*}{\partial y^*} = 0. \quad (9.1.9)$$

Since the pressure is constant in y direction at the leading order, the final system of boundary layer equations (equations for inner solution) are,

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \quad (9.1.10)$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{dp^*}{dx^*} + \frac{\partial^2 u^*}{\partial y^{*2}}. \quad (9.1.11)$$

Pressure being constant inside the boundary layer in the wall-normal direction implies that pressure variation in x can be obtained from the outer solution. Given the outer equations correspond to the ideal flow, we note that

$$p^* + \frac{|\mathbf{u}^*|^2}{2} = \text{const.} \quad (9.1.12)$$

There is a y direction velocity outside the boundary layer (as we shall see below in the solution). Intuitively, this is the wall-normal flow velocity which takes place due to the displacement of the flow by the flat plate. However, this y direction velocity is approximately $\frac{1}{\sqrt{\text{Re}_\ell}}$ times the x direction velocity, similar to boundary layer. Hence,

$$p^* + \frac{1}{2} (u_\infty^{*2} + \epsilon^2 v_\infty^{*2}) = \text{const.}, \quad (9.1.13)$$

where the subscript $()_\infty$ denotes values for $y^* \rightarrow \infty$. At leading order, we obtain,

$$\frac{dp^*}{dx^*} = -u_\infty^* \frac{du_\infty^*}{dx^*}, \quad (9.1.14)$$

which yields the complete boundary layer system of equations,

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \quad (9.1.15)$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = u_\infty^* \frac{du_\infty^*}{dx^*} + \frac{\partial^2 u^*}{\partial y^{*2}}. \quad (9.1.16)$$

The unknowns are u^* and v^* . There are two equations (one of them is nonlinear) for two variables. The boundary conditions required are,

$$u^*(y^* = 0) = 0, v^*(y^* = 0) = 0, u^*(y^* \rightarrow \infty) = u_\infty^*, u^*(x^* \rightarrow 0) = u_\infty^*. \quad (9.1.17)$$

9.2 Zero pressure gradient flat plate boundary layers

When the pressure gradient in the potential flow region outside the boundary layer is zero, the momentum equation reduces to

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{\partial^2 u^*}{\partial y^{*2}}. \quad (9.2.1)$$

We note that the boundary layer thickness is a scale determined by the flow itself. Or, there is no externally imposed length scale in the problem. Hence, we expect a self-similar solution to be valid. Dropping the (*) notation for dimensionless variables, we take the similarity ansatz as,

$$u(x, y) = g(\eta), \quad \eta = \frac{y}{\delta(x)}. \quad (9.2.2)$$

v is obtained using continuity as,

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial \eta} \frac{1}{\delta} = g' \frac{y}{\delta^2} \frac{d\delta}{dx} \implies v = \frac{d\delta}{dx} \left(\eta g - \int g(\eta) d\eta \right). \quad (9.2.3)$$

Let $\int g(\eta) d\eta = f(\eta)$, then,

$$u = f'(\eta), \quad v = (\eta f' - f) \frac{d\delta}{dx}. \quad (9.2.4)$$

Further terms are,

$$u \frac{\partial u}{\partial x} = -f' f'' \eta \frac{1}{\delta} \frac{d\delta}{dx}, \quad v \frac{\partial u}{\partial y} = f'' (\eta f' - f) \frac{1}{\delta} \frac{d\delta}{dx}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{f'''}{\delta^2}. \quad (9.2.5)$$

Substituting in the momentum equation, we obtain ,

$$f''' + \left(\delta \frac{d\delta}{dx} \right) f f'' = 0 \quad (9.2.6)$$

For the above equation to be an ODE, the term $\delta \frac{d\delta}{dx}$ should be a constant. Assuming constant C ,

$$\delta \frac{d\delta}{dx} = C \implies \delta = \sqrt{2Cx}, \quad (9.2.7)$$

and the ODE is

$$f''' + C f f'' = 0. \quad (9.2.8)$$

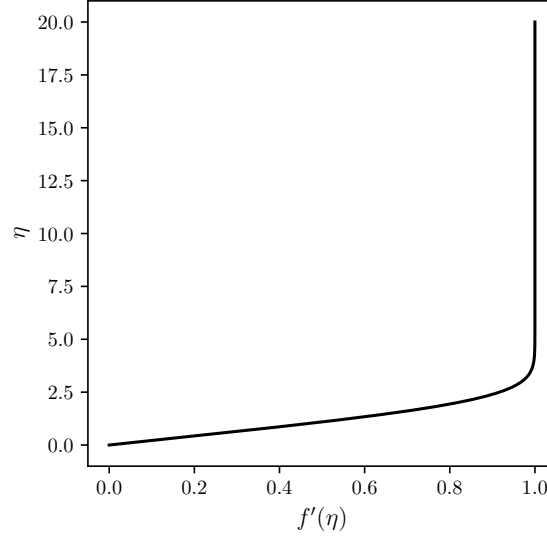


Figure 9.2: Self-similar solution of the Blasius equation obtained using $C = 1$.

This equation is known as the Blasius equation. It is a nonlinear third order ODE. To determine the required three boundary conditions, we note that $\eta \rightarrow \infty$ for both $y \rightarrow \infty$ as well as $x \rightarrow 0$. Hence, we obtain the required 3 boundary conditions

$$f'(\eta = 0) = 0, \quad f(\eta \rightarrow \infty) = 1, \quad f''(\eta = 0) = 0. \quad (9.2.9)$$

The third order ODE is usually solved using the shooting method, where a value of f'' is assumed at $\eta = 0$ and marched till large values of η and then adjusted based on the error in $\eta \rightarrow \infty$ boundary condition. It is also important to note that $f(\eta), f'(\eta), \dots$ will change with C but $u(x, y)$ remains fixed given how δ also changes with C .

9.3 Faulkner-Skan boundary layers

For boundary layers under a flow with varying streamwise velocity, we use a similarity ansatz

$$u(x, y) = u_\infty(x) f'(\eta). \quad (9.3.1)$$

Note that we are using dimensionless variables. The similarity variable $\eta = \frac{y}{\delta(x)}$. Both $u_\infty(x)$ and $\delta(x)$ are undetermined. As we will see, only under certain functional forms of these, self-similar solutions are valid. This means that only under certain type of ideal flows, boundary layers are self-similar in nature. Even though we are evaluating $u_\infty(x)$ as part of this self-similar exercise, it is important to think about this result as a constraint on $u_\infty(x)$. In actual analysis, $u_\infty(x)$ must be evaluated using the leading order outer layer equations, which are the

inviscid flow equations. Using the continuity equation, we find,

$$v = \eta u_\infty f' \frac{d\delta}{dx} - \frac{d(u_\infty \delta)}{dx} f \quad (9.3.2)$$

Substituting this in the boundary layer equation, we obtain,

$$f''' + \delta^2 \frac{du_\infty}{dx} (1 - f'^2) + \delta \frac{d(u_\infty \delta)}{dx} f f'' = 0. \quad (9.3.3)$$

This equation is an ODE in η only if $\delta^2 \frac{du_\infty}{dx}$ and $\delta \frac{d(u_\infty \delta)}{dx}$ are constants. Let,

$$\delta^2 \frac{du_\infty}{dx} = \beta, \quad \delta \frac{d(u_\infty \delta)}{dx} = \alpha. \quad (9.3.4)$$

Then,

$$\frac{d(u_\infty^2 \delta)}{dx} = \delta^2 \frac{du_\infty}{dx} + 2 \left(\delta \frac{d(u_\infty)}{dx} - \delta^2 \frac{du_\infty}{dx} \right) = 2\alpha - \beta, \quad (9.3.5)$$

which yields,

$$\delta = \sqrt{\frac{(2\alpha - \beta)x}{u_\infty}}, \quad u_\infty = x^{\frac{\beta}{2\alpha - \beta}}. \quad (9.3.6)$$

The above equations show that $\delta(x)$ is not increasing as $\sqrt{x}$ for boundary layers with freestream velocity u_∞ varying as power law of x .

In section 8.3.1, we learned that for ideal flows inside wedges of angle π/m , the flow potential is given by,

$$\phi = C_0 A r^m \cos(m\theta) \implies u_r = C_0 A m r^{m-1} \cos(m\theta), \quad u_\theta = -C_0 A m r^{m-1} \sin(m\theta). \quad (9.3.7)$$

From these, we obtain,

$$u = u_0 r^{m-1} \cos((m-1)\theta), \quad v = -u_0 r^{m-1} \sin((m-1)\theta). \quad (9.3.8)$$

For $\theta = 0$ or $\theta = \frac{\pi}{m}$ walls, we note that $u_\theta = 0$ and only tangential velocity is present. The asymptotic matching condition dictates that

$$\mathbf{u}^{\text{inviscid}}|_{\text{wall}} = \mathbf{u}^{\text{boundary layer}}|_{\text{edge}}, \quad (9.3.9)$$

where “edge” denotes the boundary layer edge or the inner layer edge, which is only achieved asymptotically with the appropriately scaled coordinate $y^* \rightarrow \infty$. For $m = 2$, we see that the potential flow is in a 90° corner. For $1 < m < 2$, the flow can be used to model boundary layer past a edge where the flow turns an

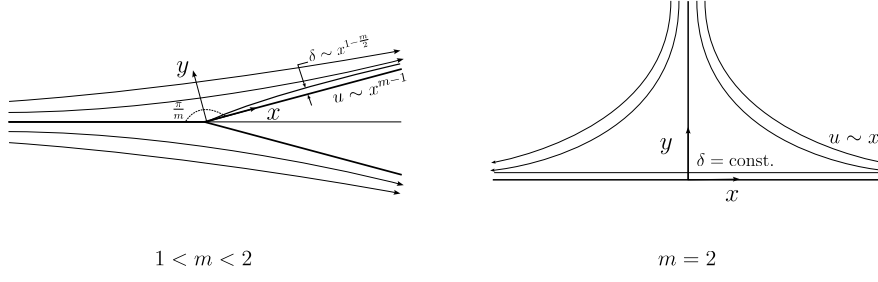


Figure 9.3: Falkner-Skan boundary layers for flows with streamwise velocity varying as power law.

angle $\frac{(m-1)\pi}{m}$. For u_∞ increasing with x , we note that δ grows slower than $\sqrt{x}$, and for decreasing u_∞ , δ increases faster than $\sqrt{x}$. For increasing u_∞ , streamwise pressure gradient is negative. At the wall, due to no-slip and hard-wall conditions, in general body attached coordinates, we can write,

$$\frac{dp}{ds} = \mu \frac{\partial^2 u}{\partial n^2} \text{ at wall.} \quad (9.3.10)$$

As $\frac{dp}{ds} < 0$, we have $\frac{\partial^2 u}{\partial n^2} < 0$. If we define shear stress as

$$\tau = \mu \frac{\partial u}{\partial n}, \quad (9.3.11)$$

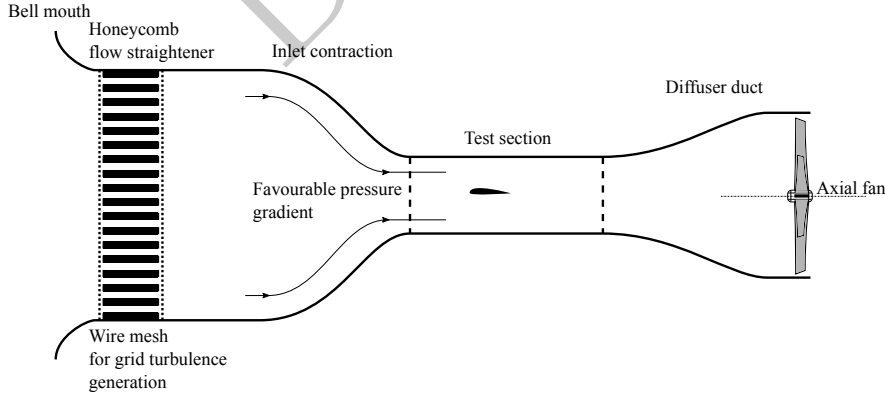


Figure 9.4: Schematic of a typical suction-tunnel (a type of an open circuit tunnel) used for aerodynamic testing at low to moderate speeds. The driven end may consist of an axial fan or a suction end of a centrifugal pump. The honeycomb flow straighteners are usually optional in a low speed rotating suction fan tunnels. The role of wire-mesh screens will become clear in subsequent chapters. The inlet contraction induces a favourable pressure gradient which suppresses boundary layer growth for maximum possible uniform flow in the test-section.

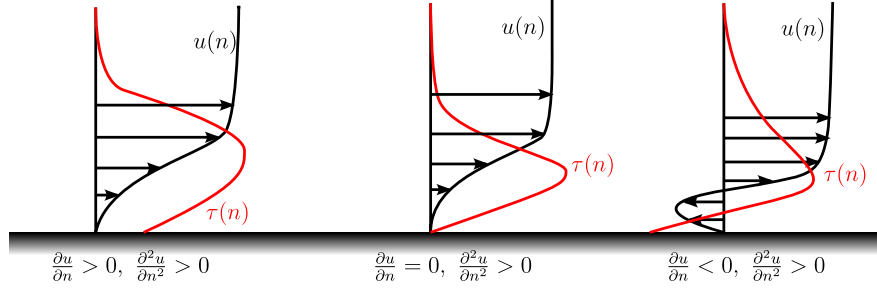


Figure 9.5: Adverse pressure gradient boundary layer profiles. The shear stress can become negative for sufficiently large adverse pressure gradient.

then $\frac{\partial \tau}{\partial n} < 0$ at wall. Since $\tau \rightarrow 0$ at the edge of the boundary layer, the shear stress increases monotonically as we approach wall from the boundary layer edge (decreasing n). Hence, the velocity profile monotonically approaches the free-stream velocity at the boundary layer edge, i.e., it is concave upwards when drawn as shown in figure 9.6. In such regions, boundary layer thickness may grow slower than $\sqrt{x}$ or even decrease with x . Contraction zones of wind-tunnels are perfect examples of application of this concept. The increasing velocity upstream of the test-section results in a decrease of boundary layer, ensuring more zone within the test-section experiences uniform (close to) flow.

On the other hand, if the pressure gradient is positive, i.e. $\frac{dp}{ds} > 0$, then $\frac{\partial^2 u}{\partial y^2} > 0$ at the wall. This means that $\frac{\partial \tau}{\partial n} > 0$ at the wall. Hence, as we move away from the wall, the shear stress increases. However, shear stress must decay as we move closer to the boundary layer edge. Hence, there is a maximum of the shear stress which must exist in the boundary layer. Such a non-monotonic behaviour of τ results in an inflection point of velocity inside the boundary layer. If the pressure gradient continues to increase further, the non-monotonic behaviour of τ can result in $\tau < 0$ at the wall. At this point, velocity has reversed in the flow, and the flow is called to have *separated from the body*. Flow separation or boundary layer separation is a very important phenomenon which is relevant in aerodynamic designs. A separated boundary layer may reattach to the body due to further decrease in pressure, or it may continue downstream generating swirling flow behind bluff bodies. At even higher Reynolds numbers, this swirling flow gives way to unsteady vortex shedding in the wake. Such vortex shedding can be observed even in cases of streamlined bodies as airfoils at sufficiently high Reynolds numbers. Due to such reasons, a streamwise increasing pressure called an adverse pressure gradient and a streamwise decreasing pressure is called a favourable pressure gradient.

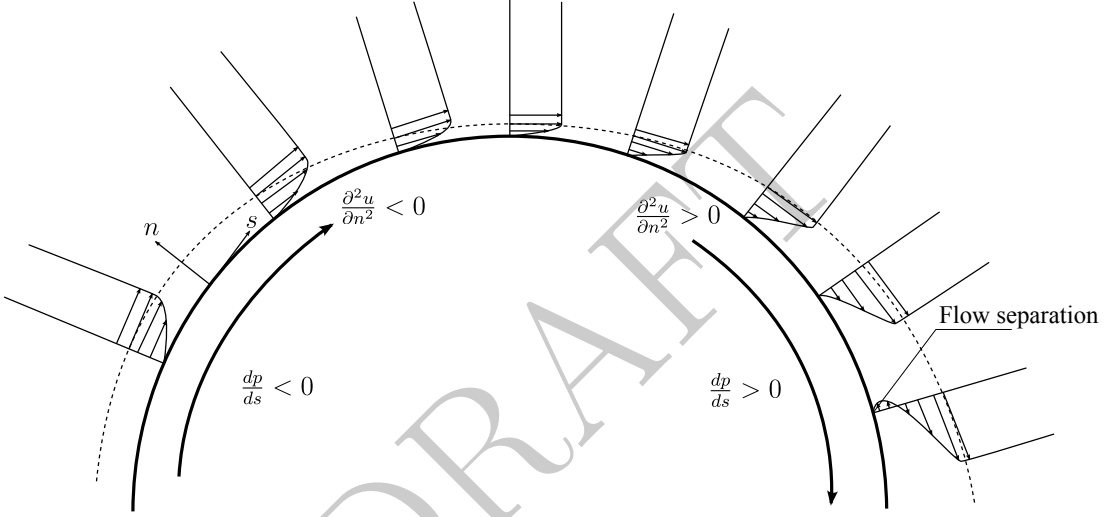
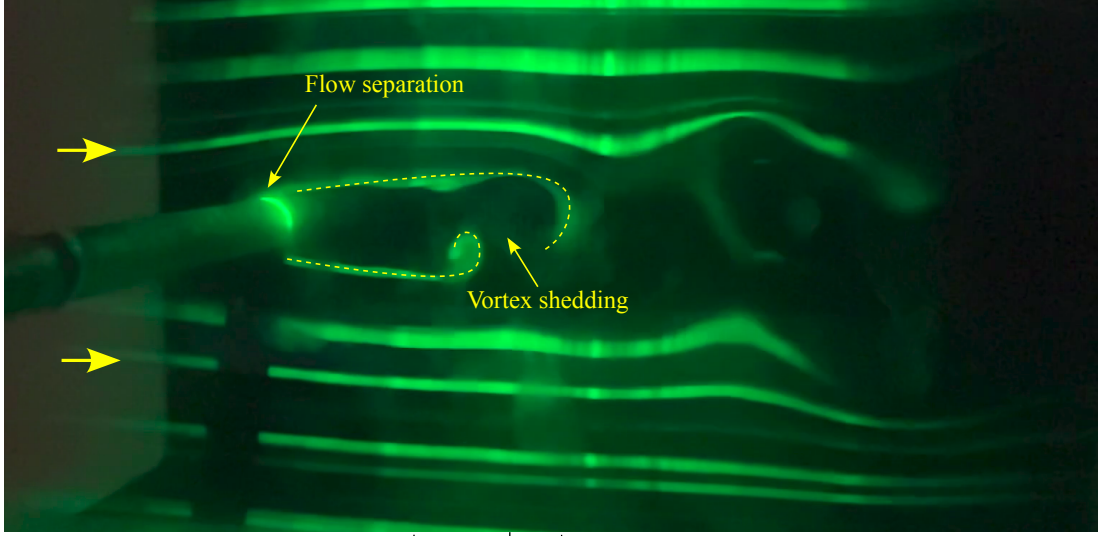


Figure 9.6: Variation of the velocity profile as the flow passes from a favourable pressure gradient to a region with adverse pressure gradient.

9.4 Application of boundary layer theory

From flat plate boundary layer theory, we notice that the streamwise velocity in dimensional form is given by (for zero pressure gradient boundary layers)

$$u(x, y) = U_{\infty} f\left(y\sqrt{\frac{U_{\infty}}{\nu x}}\right) = U_{\infty} f(\eta). \quad (9.4.1)$$

The practical quantity of interest in boundary layer theory is obviously the skin-friction drag. Shear stress at $y = 0$ is

$$\tau_w = \mu \frac{\partial u}{\partial y} \Big|_{y=0} = \mu U_{\infty} \sqrt{\frac{U_{\infty}}{\nu x}} f'(0) = \rho U_{\infty}^2 \frac{f'(0)}{\sqrt{\text{Re}_x}}, \quad (9.4.2)$$

which yields the skin-friction coefficient defined as,

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2} = \frac{2f'(0)}{\sqrt{\text{Re}_x}}. \quad (9.4.3)$$

The above expression shows that determination of $f(\eta)$ (or a good approximation) is all that is required for evaluating the practical quantity of interest – shear stress on the plate. Instead of solving the differential equation, there is another approximate way of evaluating $f(\eta)$ called the Von-Karmann - Pohlhausen method. While the method may be found in [Panton \[2024\]](#), two interesting quantities which emerge in the method are namely the *displacement thickness* δ^* and the *momentum thickness* Θ^* defined as,

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right) dy, \quad \Theta^* = \int_0^\infty \frac{u}{U_\infty} \left(1 - \frac{u}{U_\infty}\right) dy. \quad (9.4.4)$$

A simple rearrangement of the δ^* formula yields,

$$U_\infty \delta^* = \int_0^\infty (U_\infty - u) dy. \quad (9.4.5)$$

If $\delta(x)$ is the boundary layer thickness, then by definition $u = U_\infty$ for $y > \delta$. Technically, u always approaches U_∞ asymptotically. Hence, a practical value of δ is chosen by selecting the point where $u \approx 0.99U_\infty$. A boundary layer thickness defined this way is called δ_{99} . Using this, we obtain,

$$U_\infty \delta^* \approx \int_0^{\delta_{99}} (U_\infty - u) dy = U_\infty \delta_{99} - \int_0^{\delta_{99}} u dy, \quad (9.4.6)$$

or

$$\int_0^{\delta_{99}} u dy = U_\infty (\delta_{99} - \delta^*). \quad (9.4.7)$$

The above expression shows that the flux through a control volume ending at δ^* above the flat plate in a uniform flow of U_∞ is the same as flux through the control volume extending to the plate with a flat plate boundary layer on it.

9.4.1 2D jets

Besides the practical calculations possible, the boundary layer theory is also applicable in several other non-intuitive flows. Free jets are good examples of such

flows. As long as the boundary layer theory assumptions

$$\frac{\delta}{\ell} = \frac{1}{\sqrt{\text{Re}_\ell}}, \quad \text{Re}_\ell \gg 1, \quad (9.4.8)$$

hold, the Navier-Stokes equations can be reduced to the boundary layer equations. For 2D, planar incompressible flows, these equations are,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (9.4.9)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{dp}{dx} + \nu \frac{\partial^2 u}{\partial y^2}. \quad (9.4.10)$$

In free-jets, the atmospheric pressure exists outside the jets. Within the limits of boundary layer theory approximation $\frac{\delta}{\ell} = \frac{1}{\sqrt{\text{Re}_\ell}} \ll 1$, i.e., when we are sufficiently far away from the jet outlet, pressure is constant in y . Since pressure outside is atmospheric and constant in x as well, pressure is constant within the jet and we can write the momentum equation as,

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}. \quad (9.4.11)$$

Integration of the continuity equation over the highlighted control volume in red

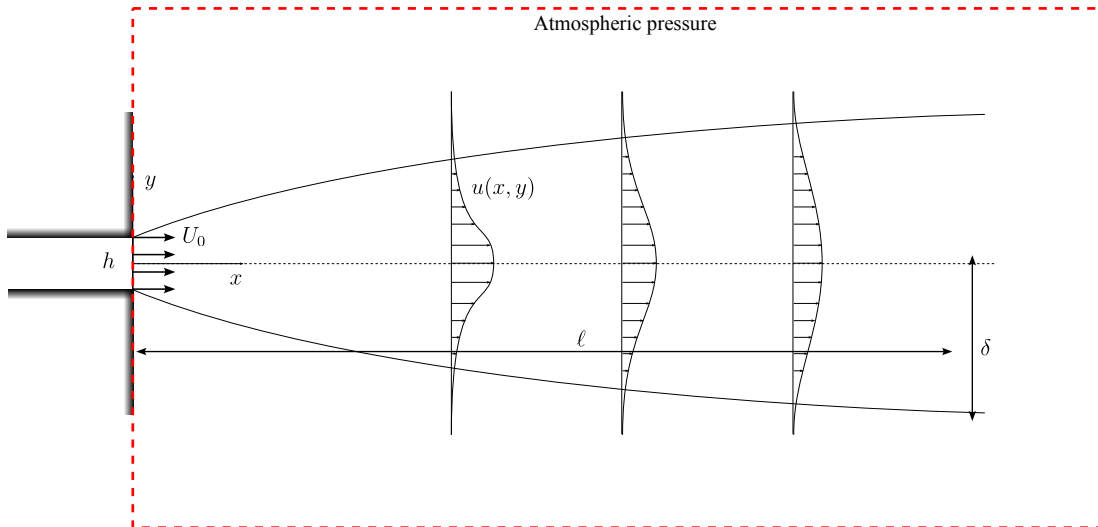


Figure 9.7: 2D planar jet

in figure 9.7 yields,

$$\int_{-H/2}^{H/2} u dy - U_0 h + 2 \int_0^x v(x, y = H) dx = 0, \quad (9.4.12)$$

where we have used the symmetry of the flow ($v(x, y) = -v(x, -y)$). The above equation shows us

$$\int_{-H/2}^{H/2} u dy = U_0 h - 2 \int_0^x v(x, y = H) dx \quad (9.4.13)$$

The quantity $\int_0^x v(x, y = H) dx$, increases for large values of x . In the limit of $H \rightarrow \infty$, the above equation yields,

$$\int_{-\infty}^{\infty} u dy = U_0 h - 2 \int_0^x v(x, y \rightarrow \infty) dx. \quad (9.4.14)$$

While the y direction velocity, however small, results in a net *entrainment of the flow* as a result of which the mass flux is actually varying in x . Below we will derive this variation exactly. Assuming $u = 0$ and $\frac{\partial u}{\partial y} = 0$ for $y \rightarrow \infty$, we obtain,

$$\int_{-\infty}^{\infty} u^2 dy = U_0^2 h = \text{const.} \quad (9.4.15)$$

With this, we proceed by checking if a self-similar solution exists. We assume the guess,

$$\psi = Ax^p f(\eta), \quad \eta = \frac{y}{Bx^q}, \quad (9.4.16)$$

where ψ is the streamfunction. The u and v velocity are obtained as,

$$u = \frac{A}{B} x^{p-q} f', \quad v = Ax^{p-1} (qf'\eta - pf). \quad (9.4.17)$$

Substituting in the boundary layer equation, we obtain,

$$f''' = \frac{ABx^{p+q-1}}{\nu} \left((p-q)f'^2 - pf f'' \right). \quad (9.4.18)$$

For a self-similar solution to exist, $p+q=1$ should hold. The other relation between p and q is through the momentum integral, which is constant. Substituting

the expression for u in the momentum integral, we obtain,

$$\frac{A^2}{B} x^{2p-q} \int_{-\infty}^{\infty} f'^2 d\eta = \text{const.} \quad (9.4.19)$$

For the above to hold $2p = q$ must hold. Hence, a self-similar solution exists if $p = 1/3$ and $q = 2/3$. Before we start solving for f , we need boundary conditions. We use the following three boundary conditions

1. $u = 0$ for $y \rightarrow \infty \implies f'(\eta \rightarrow \infty) = 0$ due to far-field
2. $\frac{\partial u}{\partial y} = 0$ for $y = 0 \implies f''(\eta = 0) = 0$ due to symmetry
3. $v = 0$ for $y = 0 \implies f(\eta = 0) = 0$ due to symmetry.

Substituting p and q , we obtain,

$$\begin{aligned} f''' + \frac{AB}{3\nu} (f'^2 + f f'') &= 0 \implies f''' + \frac{AB}{3\nu} (f f')' = 0 \implies f'' + \frac{AB}{3\nu} f f' = C_1, \\ \implies f' + \frac{AB}{6\nu} f^2 &= C_1 \eta + C_2. \end{aligned} \quad (9.4.20)$$

For $\eta \rightarrow \infty$, $f'(\eta \rightarrow \infty)$ has to be finite, so $C_1 = 0$. Hence,

$$f' = C_2 - C f^2, \quad (9.4.21)$$

where $C = \frac{AB}{6\nu}$. Solving the above, let $g = f \sqrt{\frac{C}{C_2}}$,

$$\frac{1}{\sqrt{CC_2}} \int \frac{dg}{1-g^2} = \int d\eta + K' \implies \frac{1+g}{1-g} = K e^{2\sqrt{CC_2}\eta}, \quad (9.4.22)$$

where $K = \ln K'$ is some constant. Since $f = 0$ for $\eta = 0$, we get $K = 1$ and

$$f = \sqrt{\frac{C_2}{C}} \tanh \left(\sqrt{CC_2} \eta \right). \quad (9.4.23)$$

From here, we obtain $f' = C_2 \text{sech}^2 \left(\sqrt{CC_2} \eta \right)$ from which we obtain

$$u = \frac{A}{B} x^{-1/3} C_2 \text{sech}^2 \left(\sqrt{CC_2} \eta \right). \quad (9.4.24)$$

For $\eta = 0$, we obtain $u_{\max} = \frac{A}{B} x^{-1/3} C_2$ which shows the decay of maximum

velocity in x . Substituting in the momentum integral, we obtain,

$$\int_{-\infty}^{\infty} u^2 dy = Bx^{2/3} \frac{A^2}{B^2} x^{-2/3} C_2^2 \int_{-\infty}^{\infty} \text{sech}^4 \left(\sqrt{C C_2} \eta \right) d\eta = U_0^2 h. \quad (9.4.25)$$

Upon substitution and calculating $\int_{-\infty}^{\infty} \text{sech}^4(\xi) d\xi = \frac{4}{3}$, we obtain

$$\frac{4}{3} \sqrt{6\nu} \left(\frac{AC_2}{B} \right)^{3/2} = U_0^2 h. \quad (9.4.26)$$

We note that the solution $\psi(x, y)$ can be written as,

$$\psi(x, y) = Ax^{1/3} f(\eta) = x^{1/3} \sqrt{\frac{6\nu AC_2}{B}} \tanh \left(\sqrt{\frac{AC_2}{6\nu B}} \frac{y}{x^{2/3}} \right). \quad (9.4.27)$$

Hence, individual values of A, B, C_2 do not matter, only the value of $\frac{AC_2}{B}$ matters, which can be obtained from the momentum integral. Using this solution, we can also calculate the mass flow rate and show that $\int_{-\infty}^{\infty} u dy \sim x^{1/3}$ which shows that the mass flow rate increases as $x^{1/3}$.

Boundary layer theory can be further applied to other examples as well, such as far wake region of a bluff body, shear layers (mixing layers), and boundary layers over bodies of revolution. It is important to note that the boundary layer equations are approximation of Navier-Stokes but still nonlinear. So the self-similar solutions that we have found, are only one of the solutions of these equations. There may be many more solutions. The question of *which one do we observe in practice and how often?* depends on the ability of these solutions to “resist” external perturbations, i.e., stability of these solutions. If we obtain a so-called linearly stable solution, then we can be certain of observing the actual solution in a lab (for example a lighter fluid resting on top of a heavier fluid, both with no velocity). In the next chapter, we look at this concept of linear stability and how it is used to determine stability of a flow (and its unstable modes).

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