

Practice problems : Index notation and tensor calculus

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1. A coordinate system $\bar{e}$ is obtained from the e system by a counter-clockwise rotation of 90° about the x_3 axis. The e^* coordinate system is obtained from the e system by a reflection of the x_3 axis about the $x_1 - x_2$ plane.
 - (a) Sketch the e , $\bar{e}$ and the e^* coordinate systems.
 - (b) Write $\bar{e}_i$ and e_i^* in terms of e_i .
 - (c) Determine the direction cosines, $\bar{a}_{ij}$ ($\equiv e_i \cdot \bar{e}_j$) and a_{ij}^* ($\equiv e_i \cdot e_j^*$).
 - (d) Verify that $\bar{a}_{ik}\bar{a}_{jk} = \delta_{ij}$ and $a_{ik}^* a_{jk}^* = \delta_{ij}$
 - (e) Determine the determinants of $\bar{a}_{ij}$ and a_{ij}^* .
2. With ϕ being a scalar, $\underline{u}$ a vector, and $\underline{\omega} \equiv \nabla \times \underline{u}$, show using the index notation,
 - (a) $\nabla \cdot \underline{\omega} = 0$
 - (b) $\nabla \times (\nabla \phi) = 0$
 - (c) $\nabla \times (\nabla \times \underline{u}) = \nabla(\nabla \cdot \underline{u}) - \nabla^2 \underline{u}$
 - (d) $\underline{u} \times \underline{\omega} = \frac{1}{2} (\nabla(\underline{u} \cdot \underline{u})) - (\underline{u} \cdot \nabla) \underline{u}$
3. Let $r = \sqrt{(x_k x_k)}$ denote the distance of a point x from the origin. Using the index notation find:
 - (a) $\nabla r = \partial r / \partial x_i$.
 - (b) $\nabla \left(\frac{1}{r} \right) = \partial / \partial x_i \left(\frac{1}{r} \right)$.
 - (c) $\nabla^2 \left(\frac{1}{r} \right) = \partial^2 / \partial x_i \partial x_i \left(\frac{1}{r} \right)$.
 - (d) $\nabla \times \underline{r} = \epsilon_{ijk} \frac{\partial r_k}{\partial x_j}$
4. Show that if S_{ij} are components of a symmetric second order tensor and T_{ij} has both symmetric and anti-symmetric components T_{ij}^s and T_{ij}^a respectively,
 - (a) $S_{ij} T_{ij}^a = 0$
 - (b) $S_{ij} T_{ij} = S_{ij} T_{ij}^s$
5. Consider a surface with normal unit vector $\hat{n}$. Surface gradient of a scalar ϕ is defined as,

$$\nabla_s \phi = \nabla \phi - (\hat{n} \cdot \nabla \phi) \hat{n}$$

Show that $\nabla_s \phi \cdot \hat{n} = 0$ using index notation.

6. To define force between 2 particles, each at locations $\underline{x}$ and $\underline{y}$, a potential can be defined as $V(r) = \frac{1}{4} r^4$ where $r^2 = (\underline{x} - \underline{y}) \cdot (\underline{x} - \underline{y})$ is the square of the distance between the particles. Force on each particle is given as $-\frac{\partial V}{\partial \underline{x}}$ and $-\frac{\partial V}{\partial \underline{y}}$. Simplify these expressions.

7. Using divergence theorem, show that,

$$V = \frac{1}{3} \int_S \underline{n} \cdot \underline{r} dS,$$

where S is the surface enclosing the volume V . Using above equation, find the volume of,

- (a) a rectangular parallelopiped with sides a, b and c
- (b) a right circular cone with height H and base radius R
- (c) a sphere of radius R

8. Evaluate the surface integral $\iint_S \underline{F} \cdot \underline{n} dS$ where $\underline{F} = e^x \hat{i} + e^y \hat{j} + e^z \hat{k}$ and S is the surface of the cube defined by $|x| \leq 1, |y| \leq 1$, and $|z| \leq 1$.

9. Show that area enclosed by a curve C is

$$A = \frac{1}{2} \oint_C (\underline{r} \times \underline{t}) \cdot \hat{k} dl.$$

10. According to Helmholtz' theorem, a continuous vector field $\underline{a}$ which vanishes at infinity may always be written as,

$$\underline{a} = \nabla \phi + \nabla \times \underline{A},$$

where ϕ is a scalar field and $\underline{A}$ is solenoidal ($\nabla \cdot \underline{A} = 0$).

- (a) Evaluate $\nabla \times \underline{a}$ in terms of $\underline{A}$.
- (b) Let $\underline{A}$ be written as $\underline{A} = \psi \nabla g$ and $\underline{\omega} = \nabla \times \underline{a}$ satisfy some governing equation $\nabla^2 \underline{\omega} = 0$. What is the governing equation in terms of ψ ?