

Practice problems : Flow kinematics and conservation laws

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Department of Applied Mechanics, IIT Delhi

Prof. Prateek Gupta

1. (see derivation of Reynolds Transport Theorem for a fluid blob in notes) Through this problem, we will derive the rate of change of any intensive property in an arbitrary moving control volume.

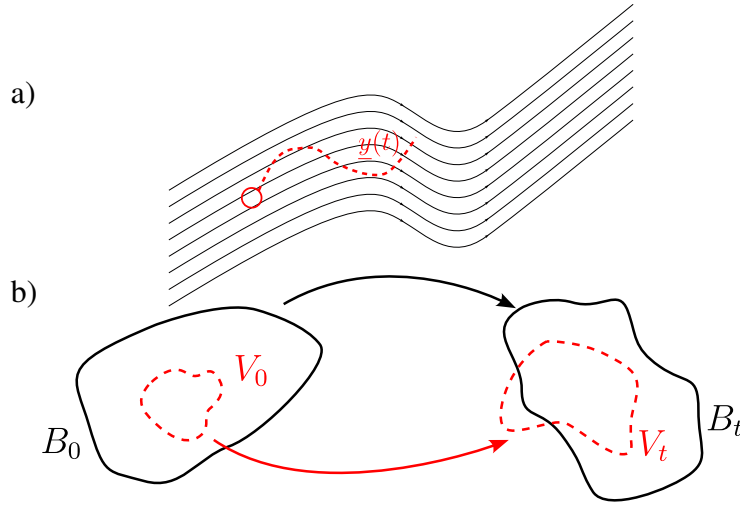


Figure 1: a) An arbitrary probe moving with coordinates $\underline{y}(t)$ in a flow. A fluid blob in initial and deforming configurations (B_0 and B_t respectively) with an arbitrary volume in reference and deforming configurations (V_0 and V_t respectively).

- (a) First, consider figure 1a). A probe which measures some intensive quantity ϕ in a flow moves with coordinate at time t given by $\underline{y}(t)$. The field measured by the probe is $\phi(\underline{y}(t), t)$. Compute the rate of change of ϕ as measured. Take velocity of probe $\dot{\underline{y}}(t) = \underline{w}$.
- (b) Now let us consider a fluid blob at instantaneous configuration B_t and some initial configuration B_0 . Also, there is an arbitrary volume V_t . The deformation and movement of V_t is our choice and is in no way connected to that of the fluid. The initial configuration of the volume V_0 is only required to compute the final result. Now, consider two mappings, $\underline{\varphi}$ and $\underline{\varphi}_V$. The mapping $\underline{\varphi}$ maps B_0 to B_t and $\underline{\varphi}_V$ maps V_0 to V_t . Let the coordinates in V_t be denoted by $\underline{y}$. We can write the total extensive quantity of the fluid inside the volume V_t as,

$$\int_{V_t} \rho \phi(\underline{y}, t) dV = \int_{V_0} \rho \phi(\underline{y}, t) J_V dV_0, \quad (1)$$

where J_V is the Jacobian of the map $\underline{\varphi}_V$. Using $\frac{dJ_V}{dt}$, show that,

$$\frac{d}{dt} \int_{V_t} \rho dV = - \int_{V_t} \nabla_{\underline{y}} \cdot (\rho(\underline{u} - \underline{w})) dV = - \oint_{S_V} \rho(\underline{u} - \underline{w}) \cdot \hat{n} dS, \quad (2)$$

where $\nabla_{\underline{y}}$ denotes differentiation w.r.t. $\underline{y}$ vector, S_V is the surface of V_t , $\underline{u}$ is the velocity of any fluid particle at $\underline{y}$ at time t , and $\underline{w}$ is the velocity of the point $\underline{y}$ due to movement/deformation of the volume V_t .

2. (See R. L. Panton, Section 5.8) The Newton's second law for control volumes is obtained by considering the intensive property $\phi = u_i$ in the RTT. Decomposing forces in the surface and volume forces, we obtain,

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j) = f_i + \frac{\partial T_{ij}}{\partial x_j}, \quad (3)$$

as obtained in class. T_{ij} is the surface stress tensor.

- (a) Consider the above equation is written at a point with position vector $\underline{r} = x_i \hat{e}_i$. Take the cross-product of the position vector with the above equation to obtain an equation for the moment of momentum $\epsilon_{ijk} x_j u_k$.
 - (b) Replace ϕ with $\epsilon_{ijk} x_j u_k + \alpha_i$ in the RTT to obtain an equation for the angular momentum.
 - (c) Show that for non-polar fluids (no moments on the molecules, no rotational acceleration $\alpha_i = 0$), $T_{ij} = T_{ji}$.
3. (See G. K. Batchelor, Section 2.3, R. L. Panton, Section 4.4, also covered in class) Consider flow velocity components u_i varying in space. Consider two points, $\underline{x}$ and $\underline{x} + \underline{r}$ separated by a vector $\underline{r}$ with very small magnitude compared to that of $\underline{x}$.

- (a) Let the velocity at $\underline{x}$ be $\underline{u}(\underline{x}) = \underline{u}$. Using Taylor series, write the expression of velocity at $\underline{u}(\underline{x} + \underline{r})$ in terms of $\underline{u}$.
- (b) Let $\underline{u}(\underline{x} + \underline{r}) - \underline{u}(\underline{x}) = \delta \underline{u}$ be the velocity difference between these two points. Decompose $\delta \underline{u}$ using symmetric and anti-symmetric parts of the velocity gradient tensor.
- (c) Assume that the symmetric part of velocity gradient is a diagonal matrix with components $\lambda_1, \lambda_2, \lambda_3$. Assuming that the anti-symmetric part of the velocity gradient tensor vanishes, compute the rate of separation of two material points on $\underline{x}$ and $\underline{x} + \underline{r}$ if $\underline{r} = (r_1, 0, 0)$.
- (d) Now assume the symmetric part of the velocity gradient tensor vanishes. Any anti-symmetric tensor Ω_{ij} can be written as,

$$\Omega_{ij} = -\frac{1}{2} \epsilon_{ijk} \omega_k, \quad (4)$$

where ω_k are the components of the *dual vector* of Ω_{ij} . Using this, show that the velocity difference between two points $\underline{x}$ and $\underline{x} + \underline{r}$ can be written as,

$$\delta \underline{u} = \frac{1}{2} \underline{\omega} \times \underline{r}, \quad (5)$$

denoting rigid body rotation with angular velocity $\frac{1}{2} \underline{\omega}$.

4. (See Kundu and Cohen, *Fluid Mechanics*, 2nd edition, Section 13 and 14, Chapter 4) Kinetic energy of a system is defined as $\Phi = \frac{1}{2} m |\underline{u}|_{cm}^2$. The corresponding intensive quantity (kinetic energy per unit mass) is $\phi = \frac{1}{2} |\underline{u}|^2$ for velocity field $\underline{u}$. The kinetic energy per unit volume is $\rho \phi$. Starting from Equation 3,
- (a) Derive the governing equation for kinetic energy per unit mass by taking a dot product of Equation 3 with $\underline{u}$.
 - (b) Using the continuity equation, derive the governing equation for kinetic energy per unit volume.
 - (c) Provide a physical interpretation of the term $\underline{u} \cdot (\nabla \cdot \underline{T})$. Consider a box-shaped fluid element. Using specific components of $\underline{u}$ and $\underline{T}$, show that the term $\underline{u} \cdot (\nabla \cdot \underline{T})$ denotes the rate of mechanical work done.
 - (d) The rate form of first law of thermodynamics is given by,

$$\frac{dE_T}{dt} = \frac{\delta Q}{\delta t} + \frac{\delta W}{\delta t}, \quad (6)$$

where δ has been used instead of d to denote infinitesimal change in path-dependent quantities. Quantity E_T is the extensive total energy given by

$$E_T = E + \frac{1}{2}m|\underline{u}|_{cm}^2, \quad (7)$$

where E is the internal energy. The corresponding intensive quantity can be written as,

$$e + \frac{1}{2}|\underline{u}|^2, \quad (8)$$

for $\underline{u}$ velocity field. Simplify the rate of work done term $(\delta W/\delta t)$ by considering forces on a control volume and using velocity at that location.

(e) Let the heat addition term $\delta Q/\delta t$ be written as

$$\frac{\delta Q}{\delta t} = - \oint_S \underline{q} \cdot \hat{n} dS. \quad (9)$$

Write the simplified differential form of the total energy equation (both per unit mass and per unit volume).

- (f) Subtract the kinetic energy (per unit volume) equation from the total energy equation (per unit volume) to derive the equation for internal energy (both per unit mass and per unit volume).
- (g) Show that the remaining term from the rate of work done contribution in the internal energy is

$$T_{ij} \frac{\partial u_i}{\partial x_j} = \underline{T} : \nabla \underline{u}. \quad (10)$$

Discuss its physical significance in a similar way as done above using a box-shaped fluid element.

(h) The rate form of the combined second law and first law of thermodynamics is given by,

$$T \frac{Ds}{Dt} = \frac{De}{Dt} - \frac{p}{\rho^2} \frac{D\rho}{Dt}, \quad (11)$$

where s is specific entropy. Using the continuity and the internal energy equation, show that the rate of increase of specific entropy for a fluid element is always positive in the absence of heat addition $\delta Q/\delta t$.

5. Consider a square shaped fluid particle in the boundary layer as shown in figure 2. The x velocity component inside the boundary layer is given by $u(x, y)$. Assume uni-directional flow.

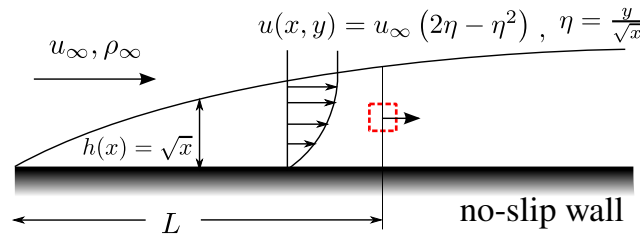


Figure 2: Deformation of a fluid particle in a boundary layer

- (a) Initial shape of the fluid parcel at $t = 0$ is shown. Sketch the shape of the red shaped fluid parcel after time δt .
- (b) Calculate the velocity gradient $\nabla \underline{u}$.
- (c) Decompose the velocity gradient in symmetric and anti-symmetric components.

6. An oscillating flow exists inside a planar channel of length $\ell = 0.1$ m and height $h = 1$ cm as shown in the figure below. The flow is uni-directional, and is given by

$$u(x, y, t) = 0.1 \sin(\omega t - kx) (1 - \cosh(10y) / \cosh(10h)),$$

where $k = 2\pi/\ell$, $\omega = 300$ rad/s, and origin is at the center of the channel. For the three fluid particles shown in the figure,

- Compute their acceleration as they pass through the coordinates shown.
- If the fluid is heated inside the channel and maintained at a spatially varying temperature $T = 300 + 50x$ K, calculate the rate of change of temperature of the fluid particles as they pass through the coordinates shown.

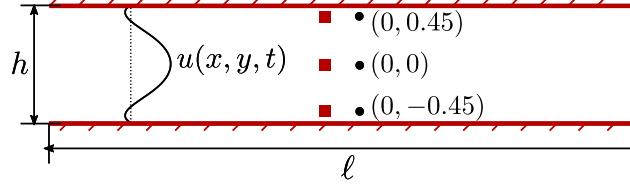


Figure 3: Oscillating flow in a channel

- For velocity field given in problem 6, calculate the acceleration of the fluid particle passing through $(0, 0)$ at time t .
- Let a function $\psi(x, y, t)$ be defined for a planar flow such that,

$$\frac{\partial \psi}{\partial x} = -v, \quad \frac{\partial \psi}{\partial y} = u.$$

Show that the iso-contour lines of this function are parallel to the streamlines of the flow at all times. Iso-contours are the lines along which the function has a constant value.

- For a flow velocity field given in cylindrical coordinates,

$$\underline{u}(r, \theta, z, t) = v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z,$$

write the expression of the acceleration of a fluid particle. Refer to handout on derivatives in curvilinear coordinates for $\underline{u} \cdot \nabla$ in cylindrical coordinates.

- The surface temperature of a lake on a winter day changes from one location to another as a function of x and y as $T(x, y) = 250 + 10xy$. If you attach a thermometer to a boat and take a path through the lake given by $x_b = t^2$ and $y_b = 2t$ where t is time, find the rate of change of temperature as recorded by the thermometer.
- For a fluid mixture consisting of N species, the mixture density is defined as $\rho = \sum_{i=1}^N \rho_i$ and the mass fraction of species i in the mixture is $y_i = \rho_i / \rho$.

- For such a mixture, we can formulate the continuity equation using the fact that the rate of change of mass of the blob of each species is zero. Show that this yields,

$$\frac{\partial \rho_i}{\partial t} + \nabla \cdot (\rho_i \underline{u}_i) = 0. \quad (12)$$

Note that i is for the i^{th} species and not for a Cartesian index.

- Sum over all species to obtain the mixture density equation as,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0. \quad (13)$$

Derive $\underline{u}$ in terms of $\underline{u}_i$.

(c) We can also write the continuity equation for each species as,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = -\nabla \cdot \underline{J}_i. \quad (14)$$

Where $\underline{J}_i$ is the mass flux of species i relative to $\underline{u}$. Write $\underline{J}_i$ in terms of $\underline{u}_i$ and $\underline{u}$.

(d) The momentum equation for species i can be written as,

$$\frac{\partial \rho \underline{u}_i}{\partial t} + \nabla \cdot (\rho \underline{u}_i \underline{u}_i) = -\nabla p_i + \nabla \cdot \underline{T}_i. \quad (15)$$

Upon summing over all species and using the definition of $\underline{u}$, show that the momentum equation for the mixture can be written as,

$$\frac{\partial \rho \underline{u}}{\partial t} + \nabla \cdot (\rho \underline{u} \underline{u}) = -\nabla p + \nabla \cdot (\underline{T} - \underline{R}), \quad (16)$$

where $\underline{T} = \sum_i \underline{T}_i$ and $\underline{R} = \rho(\underline{u}_i - \underline{u})(\underline{u}_i - \underline{u})$.