

Practice problems : Dynamical laws and exact solutions of Navier-Stokes equations

Advanced fluid mechanics

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1. (Adapted from Kundu et. al's Fluid Mechanics) A fine uniform mist of an inviscid incompressible fluid with density ρ deposits steadily at a total volume flow rate (per unit page-depth) of $2q$ on a flat surface of width $2s$ to form a liquid layer thickness $h(x)$. [Hint: it may be useful to use the sketched CV of width dx in the figure, then take $dx \rightarrow 0$; $x = 0$ is a plane of symmetry, so $u = 0$ at $x = 0$.]

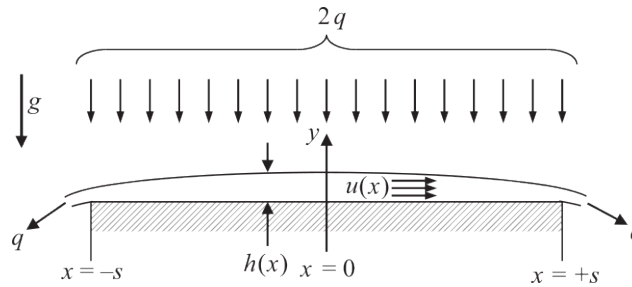


Figure 1: Mist depositing on a surface

- (a) Using the control volume, find and integrate differential equation relating $u(x)$, $h(x)$, q and s .
 - (b) Find an expression relating the fluid pressure $p_s(x)$ on the plate to $h(x)$ and other given variables.
 - (c) Find an expression for the total force on the plate as a function of the horizontal velocity.
2. The flow from a sluice gate of height h flows over an inclined plane of angle θ as shown. The height of the reservoir behind it H remains almost constant since the area of the reservoir is large compared to the sluice gate area. Compute the thickness of the flow stream down the inclined plane.

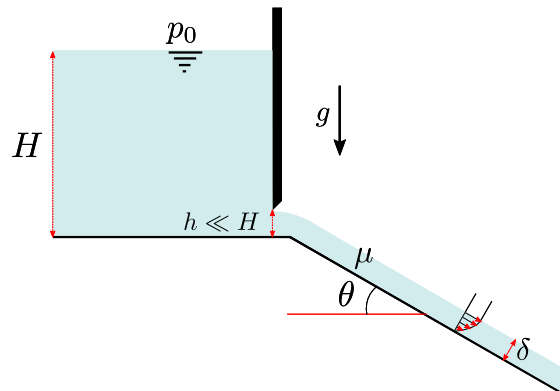


Figure 2: Sluice flow over an inclined plane

3. A fluid flows radially through a long and thin inner porous cylinder of radius R_1 , to an outer, concentric porous cylinder of radius R_2 . The non-gravitational pressure at R_1 (at the outer surface of inner cylinder) is p_1 and the total volume flow rate is Q . For steady and laminar flow between the two cylinders, simplify the governing equations (make suitable additional assumptions) to solve for pressure p in terms of p_1 and Q .

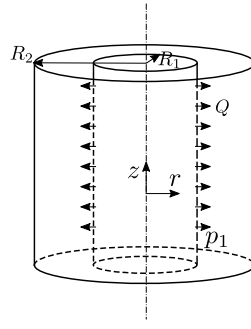
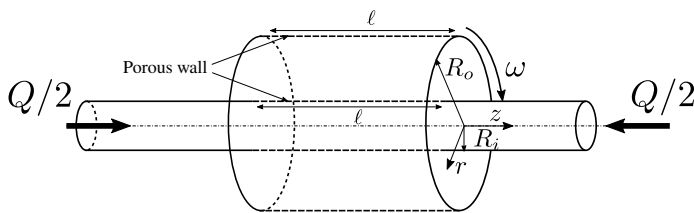
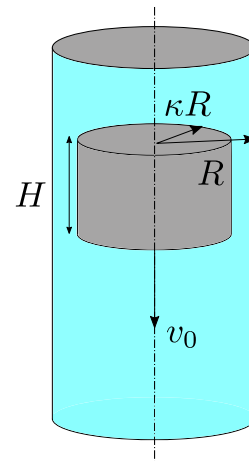


Figure 3: Radial flow through a porous cylinder

4. In a falling-cylinder viscometer, a solid cylindrical slug of radius κR falls down with a terminal velocity v_0 in a long vertical cylinder of radius R , closed at both ends. The slug's axis coincides that of cylinder at all times.
 - (a) Find the velocity profile of the fluid in the gap between the two cylinder walls $u_z(r)$ in terms of v_0 .
 - (b) Assume that the slug is of density ρ_s and the fluid is of density ρ such that $\rho_s \gg \rho$. Balance the force on the slug and find the expression of viscosity in terms of parameters that can be measured in the system.
5. A centrifuge maybe used to filter sediments from a water supply. Consider the setup shown in figure below where a water is supplied at flow rate $Q/2$ from both ends to a pipe of radius R_i which has a porous wall of length ℓ in a particular section. An outer cylinder of radius R_o and length ℓ , also made of porous wall, is rotating at a constant angular velocity ω . Clear water transpires through both the pipe and cylinder getting filtered. Assume steady incompressible axisymmetric 2D flow between the pipe and the cylinder. Assume the kinematic viscosity of the fluid as ν and density as ρ .
 - (a) Clearly write the boundary conditions for velocity components v_r and v_θ .
 - (b) Simplify the continuity equation clearly stating the assumptions. Derive an expression for v_r using the simplified continuity equation. What are the transpiration velocities at $r = R_i$ and $r = R_o$?
 - (c) Simplify θ and r momentum equations. Assume there is no pressure gradient in θ direction and flow is driven only by the rotating cylinder. Write down clearly an ODE governing pressure gradient in the radial direction and an ODE governing tangential velocity v_θ .
 - (d) Solve the ODE for v_θ . Assume that Q, ℓ , and ν are related as $Q/\ell\nu = \lambda\pi$. (Hint :- Substitute $v_\theta = Cr^n$ as a general solution.)
6. Consider a steady unidirectional pressure-driven incompressible flow $v = (0, 0, w(r))$ in a pipe of radius R of a fluid which follows the so-called ‘‘Ostwald–de Waele’’ model of a fluid such that the viscous stress



(a) A rotating filter assembly



(b) Falling cylinder viscometer

tensor is $\underline{\underline{\tau}} = 2\mu(\underline{\underline{S}})\underline{\underline{S}}$, where viscosity varies with the strain rate tensor $\underline{\underline{S}}$ as,

$$\mu(\underline{\underline{S}}) = \mu_0 \left(\frac{1}{2} \underline{\underline{S}} : \underline{\underline{S}} \right)^{\frac{n-1}{2}}.$$

- (a) Compute $\mu(\underline{\underline{S}})$ for a unidirectional flow with axial velocity only.
 - (b) Simplify the incompressible Navier-Stokes equations and find $w(r)$, using the above expression for $\mu(\underline{\underline{S}})$.
7. (Based on Panton 7.17 (B)) In class, we found an exact solution for the unsteady unidirectional flow of an incompressible fluid above a plate suddenly set into motion with constant velocity. Find a solution for the case of a plate suddenly accelerated from zero velocity. In other words, reformulate and solve the problem with a boundary condition corresponding to sudden acceleration: $u(y=0, t) = A\sqrt{\nu t}H(t)$, where $H(t)$ is the Heaviside function and A is constant. Take the similarity variable $\eta = y/2\sqrt{\nu t}$ and velocity as $u = B(\nu t)^\alpha f(\eta)$. Solve for α for which the similarity solution exists.
8. To solve for decay of ideal line vortex in time, consider the initial tangential velocity $v_\theta(r, t=0) = \frac{\Gamma}{2\pi r}$. Assuming axisymmetric conditions, solve for $v_\theta(r, t)$.
9. Consider pressure driven flow in a duct with a rectangular cross-section in z direction. The governing equation for such a case is,

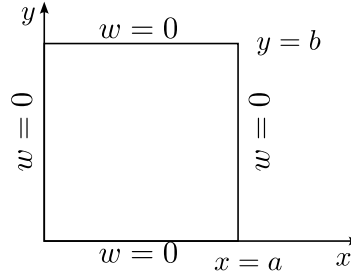


Figure 5: Cross-section of a rectangular cross-section duct

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{\mu} \frac{dp}{dz} = \text{const},$$

with boundary condition $w = 0$ on all four faces (homogeneous Dirichlet). In class, we modified w so that the modified function satisfies the Laplacian equation with inhomogeneous boundary conditions. Through this example, we will use the method of eigenfunction expansion to solve the governing Poisson equation. Take the following steps,

- (a) Consider first the homogeneous problem,

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0,$$

with homogeneous Dirichlet boundary conditions on all four faces. Find the eigenfunctions and eigenvalues of this problem. Show that the eigenfunctions of this problem are,

$$w_{mn}(x, y) = \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right).$$

- (b) Now consider the non-homogeneous problem,

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = G_0 \sin\left(\frac{M\pi x}{a}\right) \sin\left(\frac{N\pi y}{b}\right).$$

Using the eigenfunctions of the homogeneous problem, find the solution satisfying the homogeneous Dirichlet boundary conditions.

(c) Since the governing equation is linear, for a general inhomogeneity, the equation,

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = f(x, y).$$

can be considered as a linear combination of multiple problems using the decomposition of $f(x, y)$ in the eigenfunctions as,

$$f(x, y) = \sum_m \sum_n f_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right).$$

Solve the inhomogeneous Poisson equation when $f(x, y) = \text{const.}$

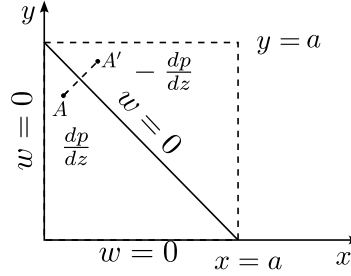


Figure 6: Cross-section of a right-isosceles triangle cross-section duct

10. Using the above problem, flow in a duct with cross-section resembling a right-isosceles triangle can be solved using superposition. For every point A in the triangle, consider an image point A' in the line $x+y=a$. If an equal and opposite pressure gradient is assumed at A' for a flow in a square cross-section duct, $w=0$ is enforced at $x+y=a$ boundary. Using the above derived solution, find the flow velocity through this duct.
11. Let us consider the start-up of the flow of fluid of viscosity μ between two walls separated by distance d and caused by a sudden impulsive movement of the bottom wall in the shown figure. The top wall is fixed and the fluid is initially at rest. The bottom wall velocity $U(t)$ is such that $U(t) = 0$ for $t \leq 0$ and $U(t) = U_0$ for $t > 0$.
 - (a) State all the boundary and initial conditions.
 - (b) Using suitable assumptions derive the PDE governing this unsteady flow in the dimensionless form. Clearly state the all the scales used for dimensionless variables.
 - (c) Solve the above equation using variable separation.
 - (d) Derive the later-time $t \rightarrow \infty$ limit of the solution. Compare it to the steady state Couette-flow solution.
12. For 2D flows, if vorticity field is given $\underline{\omega}$, the velocity field is given by,

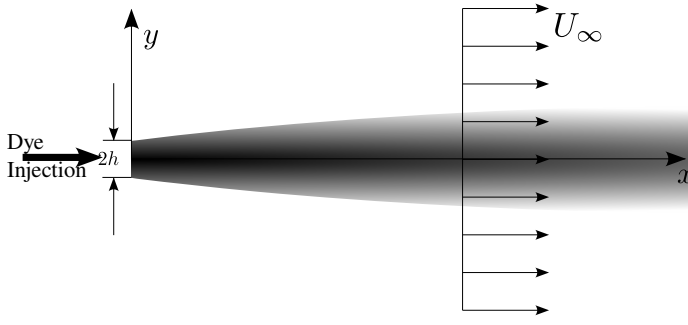
$$\underline{u}(\underline{x}) = -\frac{1}{2\pi} \nabla \times \left(\int_{\mathbb{R}^2} \ln |\underline{x} - \underline{y}| \underline{\omega}(\underline{y}) d\underline{y} \right),$$

where $\underline{y}$ and $\underline{x}$ are coordinate vectors in 2D space $\mathbb{R}^2$ and ∇ denotes gradient w.r.t $\underline{x}$. Simplify the above expression using index notation.

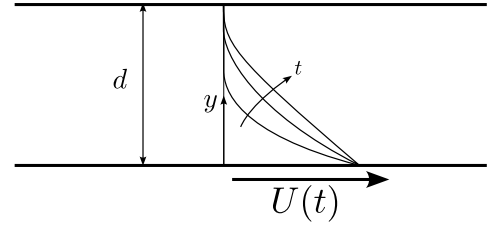
13. A dye in a flow can be modeled as a passive scalar with concentration field $c(\underline{x}, t)$ (intensive quantity). For a passive scalar with diffusivity coefficient D , the governing equation is given by,

$$\frac{\partial c}{\partial t} + \underline{u} \cdot \nabla c = D \nabla^2 c. \quad (0.0.1)$$

Consider a dye being injected very slowly at $x=0$ in a uniform, steady, parallel, and 2D flow as shown below. The dye concentration is c_0 within $-h \leq y \leq h$ at $x=0$ at all times. As the dye gets convected downstream in x , it spreads in y as shown. The value of concentration within $-h \leq y \leq h$ also changes.



Concentration field of a dye being injected in a uniform steady parallel flow.



Start-up of shear flow between two plates.

- For steady state conditions in the uniform parallel flow as shown, simplify the passive scalar equation. Ignore $\frac{\partial^2}{\partial x^2}$ term compared to $\frac{\partial^2}{\partial y^2}$
- Clearly specify all the boundary conditions on c .
- Solve the simplified PDE from part 13a. Consider Fourier-transform in y such that,

$$\hat{c}(k, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(y, x) e^{-iky} dy, \quad c(y, x) = \int_{-\infty}^{\infty} \hat{c}(k, x) e^{iky} dk.$$

Using this, derive a simple first order PDE for $\hat{c}(k, y)$ from the simplified PDE. Solve it to obtain the solution $c(x, y)$.

- An infinitely long cylinder of radius R is immersed in a Newtonian fluid. The fluid is motionless. The cylinder begins to rotate about its axis at $t = 0$ with angular velocity $\Omega \hat{e}_z$.



Figure 8: Infinitely long rotating cylinder

- Assuming axi-symmetric flow, calculate the velocity distribution at steady state by simplifying the Navier-Stokes equations in cylindrical coordinates.
- Consider the start-up flow at initial times due to cylinder rotation. In the small radius limit $R \rightarrow 0$ and $\Omega R^2 \rightarrow \frac{\Gamma}{2\pi}$ (a constant), simplify the governing equations and the boundary conditions.
- Using a similarity variable $\eta = \frac{r}{\sqrt{\nu t}}$ solve the simplified equations obtained in part 14b to obtain the unsteady velocity field. Confirm that for $t \rightarrow \infty$ this field approaches the solution in part 14a.