

Practice problems : Low Reynolds number flows and lubrication

Advanced fluid dynamics

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1. Biharmonic equation in cylindrical coordinates is given by,

$$\left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)^2 \psi(r, \theta) = 0.$$

Find the most general solutions of the form $\psi = r^m f_m(\theta)$. Leave the arbitrary coefficients as is.

2. Consider flow in a corner with a moving wall with speed U and a stationary wall at an angle α to the moving wall, as shown below

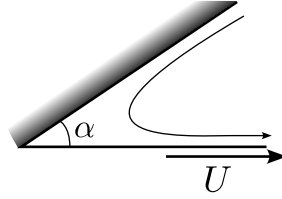


Figure 1: Taylor's scraper

- (a) Let the streamfunction $\psi(r, \theta)$ be defined in cylindrical coordinates. Using solid-wall and no-slip conditions, write the boundary conditions for ψ .
- (b) Assuming variable separation of the form $\psi = r^m f_m(\theta)$ and using the Biharmonic equation from the previous problem, find the value of m for which the above boundary conditions can be satisfied.
- (c) For the above value of m , solve for $f_m(\theta)$.
- (d) Compute the shear stress $\tau_{r\theta}$ at the bottom wall $\theta = 0$.
3. The “Stokeslet” is a fundamental solution to Stokes’ flow equations due to a point-force (Green’s function of the Stokes’ equation). The dimensionless governing equations are,

$$\nabla \cdot \mathbf{u} = 0,$$

and

$$\nabla^2 \mathbf{u} - \nabla p = \mathbf{f} \delta(\mathbf{x} - \mathbf{y}),$$

where $\delta(\mathbf{x} - \mathbf{y})$ is the Dirac delta function and force $\mathbf{f}$ is concentrated at $\mathbf{y}$. Using the definition of Fourier transform and inverse Fourier transform covered in class, and using the identities,

$$\delta(\mathbf{x} - \mathbf{y}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} d\mathbf{k}, \quad \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})}}{k^2} d\mathbf{k},$$

and

$$\frac{|\mathbf{x} - \mathbf{y}|}{8\pi} = -\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})}}{k^4} d\mathbf{k},$$

where $k = |\mathbf{k}|$, derive the velocity $\mathbf{u}$ and pressure p for a Stokeslet. Follow the procedure used in class to derive the Green's function of the Laplace equation in free space.

4. Find the pressure-drop versus flow rate relation for steady flow in a slot with gap given by $h = h_0 (1 + \epsilon \sin(kx))$ where $\epsilon \ll 1$. What is appropriate x -direction length scale?
5. Consider a sphere of radius R being pushed by an applied force F toward a solid plane boundary in a viscous fluid. At the lowest point, the gap is $b(t)$ as a function of time t .

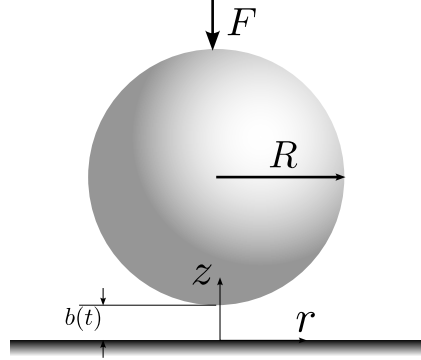


Figure 2: Sphere squeeze film

- (a) Consider the Navier-Stokes equations in cylindrical coordinates. Assuming axisymmetry, specify the condition for r/R under which the lubrication theory is valid.
- (b) Derive the Reynolds equation for this geometry. Follow the procedure from class.
- (c) Using the Reynolds equation, calculate the pressure distribution below the sphere $p(r) - p(R)$.

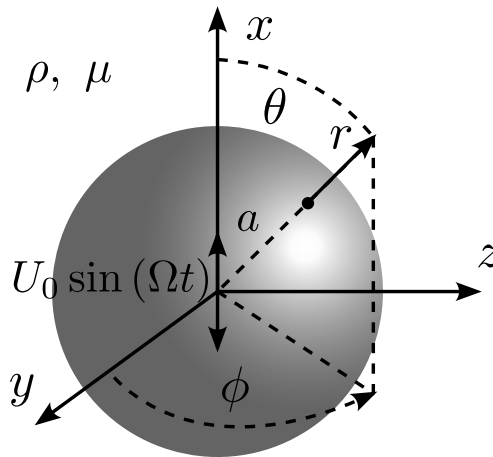


Figure 3: Vibrating sphere.

6. A sphere of radius a vibrates with velocity $U_0 \sin(\omega t) \hat{e}_x$ in a quiescent fluid of viscosity μ and density ρ .
 - (a) Non-dimensionlize the governing equations. Clearly write all the scales used.
 - (b) Under the following conditions, reduce the governing equations to those of unsteady Stokes' flow,

$$\frac{U_0 a}{\nu} \ll 1, \quad \frac{a^2 \Omega}{\nu} \sim 1.$$

- (c) Assuming axisymmetric flow (constant in ϕ), solve for the radial and tangential velocity components (u_r and u_θ respectively) at periodic steady-state. You may work with the dimensional version of the governing equation. For this, follow these steps,

- i. Assume $u_r(r, \theta, t) = \text{Re}(\hat{u}_r(r, \theta, \Omega)e^{i\Omega t})$ and similarly for $u_\theta(r, \theta, t)$, where $\text{Re}(\cdot)$ denotes the real part. Substitute in the governing equation. In the following parts, assume this variation in time to eliminate time from the governing equations.
- ii. Eliminate pressure by taking the curl of the governing equation. Write in terms of the only vorticity component $\hat{\omega}_\phi$.
- iii. Assume $\hat{u}_r = f(r) \cos \theta$ and $\hat{u}_\theta = g(r) \sin \theta$. Using the continuity equation, derive a relation between $f(r)$ and $g(r)$.
- iv. Write $\hat{\omega}_\phi$ in terms of f .
- v. Substitute the above in the governing equation for $\hat{\omega}_\phi$. Substitute $f(r) = -\frac{2}{r}h(r)$ to reduce in a simplified expression for $h(r)$. Use

$$\nabla_r^2 = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right).$$

- vi. Write the boundary conditions for $\hat{u}_r$ and $\hat{u}_\theta$ at $r \rightarrow \infty$. Using these, simplify the above equation.
- vii. Solve for $f(r)$ and $g(r)$.