

Practice problems : Asymptotic matching, inviscid, and potential flows

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1. Consider the singular perturbation problem

$$\epsilon \frac{d^2 u}{dy^2} + \frac{du}{dy} = 0, \text{ for } u(0) = 0, \text{ and } u(1) = 1,$$

where $\epsilon \ll 1$. Compute and sketch the exact, the inner, the outer, and the composite solution.

2. Through this problem, we will look at Ekman layers using matched asymptotic expansions. Ekman layers are observed in geophysical flows (close to ocean beds). Consider incompressible flow between two infinite parallel discs, which rotate coaxially at different angular speeds. The distance between the two discs is $2L$.

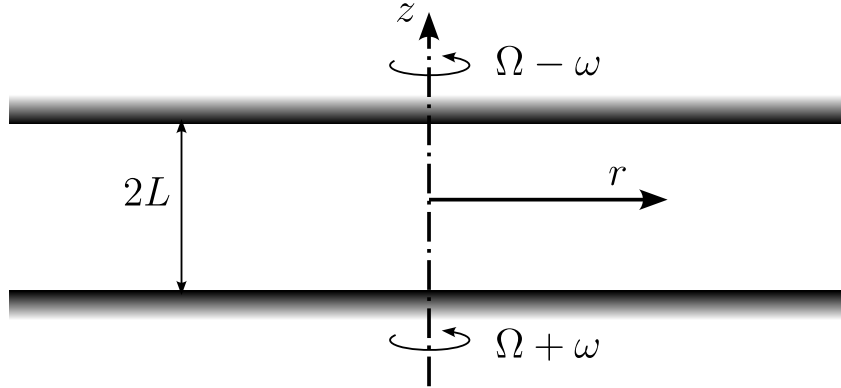


Figure 1: Ekman layer problem

- (a) Write the governing equations in cylindrical coordinates for steady 3D flow. Use the following ansatz for velocity components and pressure ,

$$u_r = r\omega U(Z), \quad u_\theta = r\Omega + r\omega V(Z), \quad u_z = L\omega W(Z),$$

and

$$p = \frac{1}{2}\rho\Omega^2 r^2 + \frac{1}{2}\rho\Omega\omega r^2 A + \rho\Omega\omega L^2 B(Z).$$

Substitute the above in the Navier-Stokes and the continuity equations to obtain dimensionless equations for U, V, W and A, B . Identify the two dimensionless parameters $Ro = \frac{\omega}{\Omega}$ (Rossby number) and $Ek = \frac{\nu}{\Omega L^2}$ (Ekman number).

- (b) Specify the boundary conditions on U, V, W .
- (c) Show that in the limit of small Rossby number $Ro \ll 1$, the above dimensionless nonlinear ODEs reduce to linear ODEs. Note that in Ro , the problem is not singular so a Naïve substitution is O.K. to get the leading order solution.

(d) Expand the unknowns in power of Ek as follows,

$$\begin{pmatrix} A \\ U \\ V \\ W \end{pmatrix} = \begin{pmatrix} A_0 \\ U_0 \\ V_0 \\ W_0 \end{pmatrix} + Ek^\alpha \begin{pmatrix} A_1 \\ U_1 \\ V_1 \\ W_1 \end{pmatrix} + \dots$$

Substitute in the linear ODEs obtained above and solve A_0, U_0, V_0, W_0 and A_1, U_1, V_1, W_1 . Can these solutions satisfy the boundary conditions? Write the solutions in terms of u_r, u_θ , and u_z to visualize this as a ‘core’ solution.

(e) To find the solution close to the discs which satisfies the no-slip conditions, we have to solve for bottom and top disc separately. Introduce local variables,

$$\eta_\pm = \frac{1 \mp Z}{Ek^\beta}$$

for some exponent β . Using U, V, W expansion as,

$$\begin{pmatrix} U(\eta_\pm) \\ V(\eta_\pm) \\ W(\eta_\pm) \end{pmatrix} = \begin{pmatrix} U_0(\eta_\pm) \\ V_0(\eta_\pm) \\ W_0(\eta_\pm) \end{pmatrix} + Ek^\beta \begin{pmatrix} U_1(\eta_\pm) \\ V_1(\eta_\pm) \\ W_1(\eta_\pm) \end{pmatrix} + \dots$$

determine β for which the ODEs are non-trivial in the limit $Ek \rightarrow 0$.

(f) Specify the appropriate boundary conditions on $U_0(\eta_\pm), V_0(\eta_\pm), W_0(\eta_\pm)$ and $U_1(\eta_\pm), V_1(\eta_\pm), W_1(\eta_\pm)$.

(g) Solve for a composite solution using asymptotic matching.

3. For a superposition of a free stream and a vortex pair at $(0, \pm a)$ of circulation $\mp\Gamma$, obtain the stream function, velocity potential, and the location of stagnation points along the x axis using the complex potential.

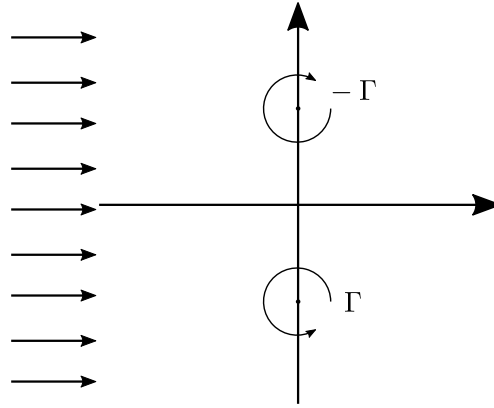


Figure 2: Vortex pair in a uniform stream

4. For a 2D flow, the velocity potential function ϕ is given by $\phi = x^2 - y^2$.

(a) Determine the velocity components in x and y directions.

(b) Show that velocity components satisfy flow continuity and irrotationality.

(c) Determine the stream function

(d) Sketch the constant ϕ and ψ lines.

(e) Uniform flow past a vortex closed to a wall can be solved using this problem. Apply *method of images* to it to convert into a situation shown in above figure. Assume the plate is at $y = 0$. Calculate the total upward force on the plate (per unit length).

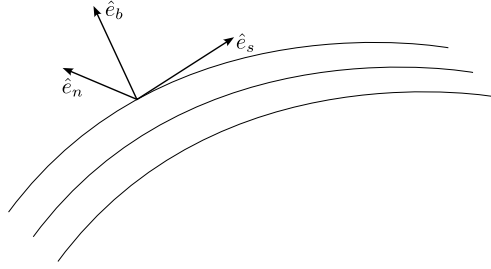


Figure 3: Streamline coordinates

5. For an ideal vortex, circumferential velocity u_θ is defined as $\Gamma/2\pi r$. However, such a velocity field is difficult to use in models due to very large velocity near $r \rightarrow 0$. To mitigate this problem, a Rankine vortex is used for which the tangential velocity is defined as,

$$u_\theta = \frac{\Gamma}{2\pi} \begin{cases} r/a^2 & r \leq a \\ 1/r & r > a \end{cases}.$$

Assuming no radial velocity exists, is the above flow model irrotational? If not, calculate the vorticity field. Calculate the streamfunction and velocity potential for this model.

6. For a flow past a rotating cylinder of radius a , the complex velocity potential for a uniform flow, a doublet, and an ideal vortex can be added.
- Using the superposed complex potential, calculate u_r and u_θ . Relate the circulation, Γ with the angular speed Ω of the cylinder.
 - Determine the location of stagnation points on the cylinder surface (where velocity vanishes).
 - Using Bernoulli's equation, determine the pressure at the surface of the cylinder given that pressure far upstream is p_∞ .
 - Integrate the pressure to calculate the net lift and drag force on the cylinder.
7. Using the method of images, find the velocity potential and stream function due to a source of strength m at a distance a from the wall in $+x$ direction.
8. In the limit $Re \rightarrow \infty$, the Navier-Stokes equations reduce to the Euler's equations for inviscid flow without any rescaling of length scales. Euler's equations can be derived in the streamline coordinates system with tangential unit vector $\hat{e}_s$, normal unit vector $\hat{e}_n$, and bi-normal unit vector $\hat{e}_b = \hat{e}_s \times \hat{e}_n$.
- Write the Euler's equations in the coordinate independent form.
 - Show that the convective term can be written as,

$$\mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{u} \frac{\partial \mathbf{u}}{\partial s} - \frac{u^2}{R} \hat{e}_n,$$

where R is the local radius of curvature of the streamline and u is the speed.

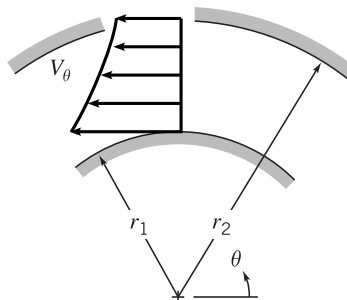


Figure 4: Circular bend

(c) Decomposing the pressure gradient term, derive the Euler's equations in the streamline coordinates.

Note that the definition of ∇ is $\nabla = \hat{e}_s \frac{\partial}{\partial s} + \hat{e}_n \frac{\partial}{\partial n} + \hat{e}_b \frac{\partial}{\partial b}$.

9. Assuming variation of tangential velocity as $V_\theta = C/r$ and inviscid flow, compute the flow rate through a duct bent in a circular arc with inner radius r_1 and outer radius r_2 in terms of pressure difference $\Delta p = p_2 - p_1$ at the two walls. Assume unit width of the duct in the page.
10. (From Kundu et al.) There is a point source of strength Q (m^3/s) at the origin, and a uniform line sink of strength $k = Q/a$ extending from $x = 0$ to $x = a$. The two are combined with a uniform stream U parallel to x axis. Show that the combination represents the flow past a closed surface of revolution of airship shape. Find the length of this shape.