

APL361 (2025-2026-I): Quiz-1

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Maximum score is 40. There are 3 questions in total. Time limit is 60 minutes. Clearly write all your assumptions and steps. No marks will be given for correct answers without any steps.

1. Evaluate/Prove using index notation (no points for expanding individual summation terms and calculating/brute force), **(14)**

(a) $\delta_{ij}\delta_{ij}$ **Answer:-** $\delta_{ij}\delta_{ij} = \delta_{ii} = 3$ **(1)**

(b) $\epsilon_{ijk}\epsilon_{ijk}$ **Answer:-** $\epsilon_{ijk}\epsilon_{ijk} = \delta_{jj}\delta_{kk} - \delta_{jk}\delta_{jk} = 3 \times 3 - 3 = 6$ **(2)**

(c) $\underline{u} \cdot \nabla \underline{u} = \nabla \left(\frac{|\underline{u}|^2}{2} \right) - \underline{u} \times \underline{\omega}$ where $\underline{\omega} = \nabla \times \underline{u}$

Answer:- $\underline{u} \times \underline{\omega} = \epsilon_{ijk}\epsilon_{klm}u_j \frac{\partial u_m}{\partial x_l} \hat{e}_i = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})u_j \frac{\partial u_m}{\partial x_l} \hat{e}_i =$
 $u_j \frac{\partial u_j}{\partial x_i} \hat{e}_i - u_j \frac{\partial u_i}{\partial x_j} \hat{e}_i = \nabla \left(\frac{|\underline{u}|^2}{2} \right) - \underline{u} \cdot \nabla \underline{u}$ **(3)**

(d) Determinant of a second order tensor $\underline{T}$ and components of it's inverse can be written as,

$$\det(\underline{T}) = \frac{1}{6}\epsilon_{ijk}\epsilon_{lmn}T_{li}T_{mj}T_{nk}, \text{ and } [T^{-1}]_{ij} = \frac{1}{2\det(\underline{T})}\epsilon_{jpq}\epsilon_{ikl}T_{pk}T_{ql}.$$

Show that $\frac{\partial(\det(\underline{T}))}{\partial T_{ij}} = \det(\underline{T})[T^{-1}]_{ji}$. **(4)**

Answer:- $\frac{\partial(\det(\underline{T}))}{\partial T_{ij}} = \frac{1}{6}\epsilon_{pqr}\epsilon_{lmn}(T_{mq}T_{nr}\delta_{li}\delta_{pj} + T_{lp}T_{nr}\delta_{im}\delta_{qj} + T_{lp}T_{mq}\delta_{in}\delta_{rj})$
 $= \frac{1}{6}(\epsilon_{jqr}\epsilon_{imn}T_{mq}T_{nr} + \epsilon_{pjr}\epsilon_{lin}T_{lp}T_{nr} + \epsilon_{pqj}\epsilon_{lmi}T_{lp}T_{mq})$
 $= \frac{1}{6}(\epsilon_{jqr}\epsilon_{imn}T_{mq}T_{nr} + \epsilon_{jrp}\epsilon_{inl}T_{lp}T_{nr} + \epsilon_{jpq}\epsilon_{ilm}T_{lp}T_{mq})$
 $= \frac{1}{6}(\epsilon_{jqr}\epsilon_{imn}T_{mq}T_{nr} + \epsilon_{jqr}\epsilon_{imn}T_{nr}T_{mq} + \epsilon_{jqr}\epsilon_{imn}T_{mq}T_{nr}) = \frac{1}{2}\epsilon_{imn}\epsilon_{jqr}T_{mq}T_{nr}$

(e) Let $\underline{r} = r_i\hat{e}_i = x_i\hat{e}_i$ be the position vector of a point, with magnitude r . Simplify, $\nabla^2(r^p)$. **(4)**

Answer:- $\nabla^2 r^p = \nabla \cdot (\nabla r^p) = p\nabla \cdot (r^{p-1}\nabla r) = p\nabla \cdot \left(r^{p-1} \frac{\underline{r}}{r} \right) = p\nabla \cdot (r^{p-2}\underline{r})$
 $= p((p-2)r^{p-3}\underline{r} \cdot \nabla r + r^{p-2}\nabla \cdot \underline{r}) = p\left((p-2)r^{p-3}\frac{r^2}{r} + 3r^{p-2}\right) = p(p+1)r^{p-2}$

2. Electrohydrodynamics is the study of electric field driven fluid flow of a dielectric fluid (fluid with bulk charge density). For a steady electric field $\underline{E}$ on a dielectric fluid of permittivity ϵ , the Coulombic force per unit volume is given by $\underline{F}_e = \underline{E}\nabla \cdot (\epsilon \underline{E}) - \frac{1}{2}|\underline{E}|^2\nabla\epsilon$. Steady electric field implies $\nabla \times \underline{E} = 0$. Show that the force $\underline{F}_e$ can be written as,

$$F_{ei} = \frac{\partial T_{ij}}{\partial x_j}.$$

Determine T_{ij} tensor. (Hint:- Start by simplifying $\epsilon \underline{E} \times (\nabla \times \underline{E})$). **(12)**

Answer:- Consider $\epsilon \underline{E} \times (\nabla \times \underline{E}) = \epsilon_{ijk}\epsilon_{klm}\epsilon E_j \frac{\partial E_m}{\partial x_l} \hat{e}_i = \epsilon E_j \frac{\partial E_j}{\partial x_i} \hat{e}_i - \epsilon E_j \frac{\partial E_i}{\partial x_j} \hat{e}_i$

$$0 = \varepsilon \nabla \left(\frac{|\underline{E}|^2}{2} \right) - \varepsilon \underline{E} \cdot \nabla \underline{E} = \varepsilon \nabla \left(\frac{|\underline{E}|^2}{2} \right) - \nabla \cdot (\varepsilon \underline{E} \otimes \underline{E}) + \underline{E} \nabla \cdot (\varepsilon \underline{E})$$

$$\underline{E} \nabla \cdot (\varepsilon \underline{E}) = \nabla \cdot (\varepsilon \underline{E} \otimes \underline{E}) - \varepsilon \nabla \left(\frac{|\underline{E}|^2}{2} \right)$$

Hence, $\underline{E} \nabla \cdot (\varepsilon \underline{E}) - \frac{1}{2} |\underline{E}|^2 \nabla \varepsilon = \nabla \cdot (\varepsilon \underline{E} \otimes \underline{E}) - \nabla \left(\varepsilon \frac{|\underline{E}|^2}{2} \right) = \nabla \cdot \left(\varepsilon \underline{E} \otimes \underline{E} - \frac{\varepsilon |\underline{E}|^2}{2} \underline{I} \right)$. We obtain the expression,

$$T_{ij} = \varepsilon E_i E_j - \frac{\varepsilon |\underline{E}|^2}{2} \delta_{ij}.$$

3. The rate of deformation work done is given by $\phi = T_{ij} \frac{\partial u_i}{\partial x_j}$, where $\underline{T}$ is the stress tensor and $\underline{u}$ is velocity **(10)**.

- (a) Show that $\phi = T_{ij} S_{ij}$ ($\underline{S}$ is the strain rate tensor) **(2)**.

Answer:- $\phi = T_{ij} \frac{\partial u_i}{\partial x_j} = T_{ij} (S_{ij} + \Omega_{ij})$. Ω_{ij} is anti-symmetric. $T_{ij} \Omega_{ij} = T_{ji} \Omega_{ji} = -T_{ij} \Omega_{ij} = 0$.

- (b) For a particular fluid, the stress tensor is given by $T_{ij} = -p \delta_{ij} + 2\mu S_{ij} - \frac{2}{3} \mu (\nabla \cdot \underline{u}) \delta_{ij}$, where p is a field and μ can be taken as constant. Show that $\phi + p \nabla \cdot \underline{u} \geq 0$. (Hint:- Think of type of quantities which are always ≥ 0) **(12)**

Answer:- $\phi = T_{ij} S_{ij} = -p \delta_{ij} S_{ij} + 2\mu S_{ij} S_{ij} - \frac{2\mu}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} S_{ij}$.

$\phi + p \nabla \cdot \underline{u} = 2\mu S_{ij} S_{ij} - \frac{2\mu}{3} (\nabla \cdot \underline{u})^2 = 2\mu \left(S_{ij}^2 + \frac{1}{3} (\nabla \cdot \underline{u})^2 - \frac{2}{3} (\nabla \cdot \underline{u})^2 \right)$. Since $\nabla \cdot \underline{u} = S_{ii}$,

$\phi + p \nabla \cdot \underline{u} = 2\mu \left(S_{ij}^2 + (\nabla \cdot \underline{u})^2 \frac{\delta_{ij} \delta_{ij}}{3^2} - \frac{2}{3} S_{ij} \nabla \cdot \underline{u} \delta_{ij} \right) = 2\mu \left(S_{ij} - \frac{1}{3} \nabla \cdot \underline{u} \delta_{ij} \right)^2 \geq 0$