

# APL361 (2025-2026-I): Quiz-2

Department of Applied Mechanics, IIT Delhi

August 29, 2024

Maximum score is 40. There are 2 questions in total. Time limit is 60 minutes. Clearly write all your assumptions and steps. No marks will be given for correct answers without any steps.

1. Electrohydrodynamics is the study of electric field driven fluid flow of a dielectric fluid (fluid with bulk charge density). For a steady electric field  $\underline{E}$  on a dielectric fluid of permittivity  $\varepsilon$ , the Coulombic force per unit volume is given by  $\underline{F}_e = \underline{E} \nabla \cdot (\varepsilon \underline{E}) - \frac{1}{2} |\underline{E}|^2 \nabla \varepsilon$ . Steady electric field implies  $\nabla \times \underline{E} = 0$ .

For an interface between two dielectric Newtonian isotropic fluids of permittivity, viscosity, and density  $\varepsilon_1, \mu_1, \rho_1$  and  $\varepsilon_2, \mu_2, \rho_2$ , respectively, with uniform surface tension coefficient  $\sigma$ , derive the dynamic interface condition. Assume incompressible flow **(10)**

Answer:- The Coulombic force can be written as the divergence of a tensor as following.

$$\text{Consider } \varepsilon \underline{E} \times (\nabla \times \underline{E}) = \epsilon_{ijk} \epsilon_{klm} \varepsilon E_j \frac{\partial E_m}{\partial x_l} \hat{e}_i = \varepsilon E_j \frac{\partial E_j}{\partial x_i} \hat{e}_i - \varepsilon E_j \frac{\partial E_i}{\partial x_j} \hat{e}_i$$

$$0 = \varepsilon \nabla \cdot \left( \frac{|\underline{E}|^2}{2} \right) - \varepsilon \underline{E} \cdot \nabla \underline{E} = \varepsilon \nabla \cdot \left( \frac{|\underline{E}|^2}{2} \right) - \nabla \cdot (\varepsilon \underline{E} \otimes \underline{E}) + \underline{E} \nabla \cdot (\varepsilon \underline{E})$$

$$\underline{E} \nabla \cdot (\varepsilon \underline{E}) = \nabla \cdot (\varepsilon \underline{E} \otimes \underline{E}) - \varepsilon \nabla \cdot \left( \frac{|\underline{E}|^2}{2} \right)$$

$$\text{Hence, } \underline{E} \nabla \cdot (\varepsilon \underline{E}) - \frac{1}{2} |\underline{E}|^2 \nabla \varepsilon = \nabla \cdot (\varepsilon \underline{E} \otimes \underline{E}) - \nabla \cdot \left( \varepsilon \frac{|\underline{E}|^2}{2} \underline{I} \right) = \nabla \cdot \left( \varepsilon \underline{E} \otimes \underline{E} - \frac{\varepsilon |\underline{E}|^2}{2} \underline{I} \right).$$

From discussion in class, we know the dynamic interface condition is

$$\underline{T} \cdot \hat{n} \Big|_2 - \underline{T} \cdot \hat{n} \Big|_1 = 2\mathcal{H}\sigma\hat{n} \quad (1)$$

The control volume derivation of the above result carries +5 marks. The total stress tensor components  $T_{ij}$  can be written as,

$$T_{ij} = - \left( p + \frac{\varepsilon |\underline{E}|^2}{2} \right) \delta_{ij} + 2\mu S_{ij} + \varepsilon E_i E_j. \quad (2)$$

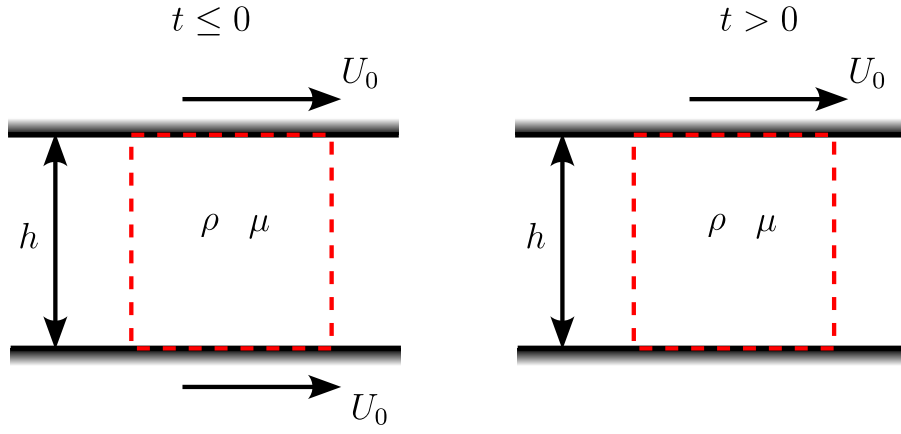
2. Two infinitely long plates separated by a gap  $h$  are moving with constant speed  $U_0$  as shown. The plates have been moving for a very long time. At an instant (end of time  $t = 0$ ), the bottom plate is stopped and brought to rest impulsively. For  $t > 0$ , the top plate continues to move with bottom plate at rest. Assume 2D planar flow at all times and that the plates are hard walls.

- (a) Show that the flow is uni-directional, given the plates are infinitely long. **(1)**

Answer:- Since the flow is 2D planar, the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad (3)$$

Since the plates are infinitely long,  $\frac{\partial(u, v)}{\partial x} = 0 \implies v = \text{const.}$ . Due to hard walls, we get  $v = 0$ . Hence the flow is unidirectional.



Problem 2

- (b) Since the plates have been moving for a very long time before the bottom plate stops, the flow between them is steady. Compute this steady solution. **(2)**

Answer:- Since the flow is unidirectional with infinite plates, the  $y$  momentum equation yields  $\frac{\partial p}{\partial y} = 0$ . The  $x$  momentum equation yields,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}. \quad (4)$$

At steady state, we obtain uniform flow  $u = U_0$ .

- (c) Compute the unsteady solution after  $t > 0$ . **(20)**

Answer:- The boundary conditions and initial conditions are,

$$u(y = h, t) = U_0, u(y = 0, t) = 0, u(y, t = 0) = U_0. \quad (5)$$

Solving the system with separation of variables, we get the general solution,

$$u(y, t) = A_0 + B_0 y + \sum_k e^{-\nu k^2 t} (A_k \sin(ky) + B_k \cos(ky)). \quad (6)$$

Using the boundary conditions, we obtain  $B_k = 0, A_0 = 0, B_0 = \frac{U_0}{h}$  and the eigenvalues,

$$k_n = \pm \frac{n\pi}{h}. \quad (7)$$

Substituting, we obtain the solution,

$$u(y, t) = \frac{U_0 y}{h} + \sum_{n=1}^{\infty} \tilde{A}_n e^{-\nu k_n^2 t} \sin(k_n y). \quad (8)$$

Note that we can sum from  $-\infty$  to  $\infty$  and then combine the terms to obtain the above expression as ,

$$\sum_{n=-\infty, n \neq 0}^{\infty} A_n e^{-\nu k_n^2 t} \sin(k_n y) = \sum_{n=1}^{\infty} (A_n - A_{-n}) e^{-\nu k_n^2 t} \sin(k_n y) = \sum_{n=1}^{\infty} \tilde{A}_n e^{-\nu k_n^2 t} \sin(k_n y). \quad (9)$$

The initial condition yields,

$$\sum_{n=1}^{\infty} \tilde{A}_n \sin(k_n y) = U_0 \left(1 - \frac{y}{h}\right). \quad (10)$$

Using the orthogonality, we can show that  $\int_0^h \sin(k_m y) \sin(k_n y) dy = \frac{h}{2} \delta_{mn}$ . Hence,

$$\tilde{A}_m \frac{h}{2} = U_0 \int_0^h \left(1 - \frac{y}{h}\right) \sin(k_m y) dy = \frac{U_0}{k_m} \quad (11)$$

which yields

$$u(y, t) = \frac{U_0 y}{h} + \frac{2U_0}{\pi} \sum_{n=1}^{\infty} \frac{e^{-\nu k_n^2 t} \sin(k_n y)}{n} \quad (12)$$

(d) What is the steady solution for  $t > 0$ ? **(1)**

Answer:- At  $t \rightarrow \infty$ , we obtain

$$u(y) = \frac{U_0 y}{h}. \quad (13)$$

(e) What is the change in total vorticity ( $\underline{\omega} = \nabla \times \underline{u}$ ) inside the square shown by red dashed lines as the bottom plate stops? Compute this change between the two steady state solutions. Justify this change in vorticity. **(3)**

Answer:- For the current situation  $\underline{\omega} = -\frac{\partial u}{\partial y} \hat{k}$ . For  $t < 0$ , since the flow is uniform,  $\underline{\omega} = 0$  and so is the vorticity inside the square. For  $t \rightarrow \infty$ ,  $\underline{\omega} = -\frac{U_0}{h} \hat{k}$ . Hence the change in vorticity is  $-U_0 h \hat{k}$ .

Stopping the bottom wall introduces vorticity in  $-\hat{k}$  direction at the bottom wall which diffuses upwards.

(f) Sketch the variation of vorticity in time. Show the initial, final, and atleast two intermediate states. **(3)**

