

APL361 (2025-2026-I): Quiz-3

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Maximum score is 40. There is only 1 problem. Time limit is 60 minutes. Clearly write all your assumptions and steps. No marks will be given for correct answers without any steps.

Important relations

- Axisymmetric incompressible flow in spherical coordinates :

$$u_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}, u_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r},$$

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right)$$

$$\nabla^2 \mathbf{u} = \left(\frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u_r}{\partial \theta} \right) - \frac{2}{r^2 \sin \theta} \frac{\partial (u_\theta \sin \theta)}{\partial \theta} \right) \hat{e}_r + \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (u_\theta \sin \theta) \right) \right) \hat{e}_\theta$$

- For ODE : $(1 - \eta^2)f'''' - 4\eta f''' + 4f'' = 0$, consider two solutions $f = \eta^3$ and $f = \eta$.

Problem

1. Consider incompressible flow in a cone with polar angle θ_0 due to a source of constant flow rate Q_0 at the apex (corner) of the cone. Assume purely radial flow. We will solve for the flow away from the apex.

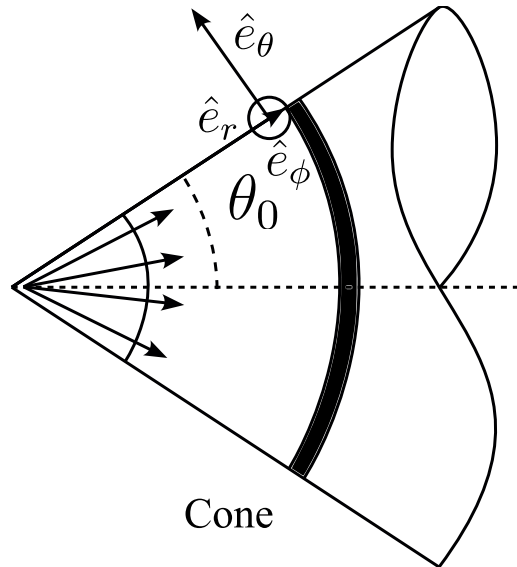


Figure 1: Flow due to a source at the apex of a cone

- (a) If constant flow rate Q_0 is maintained, then what is the form of the streamfunction?

Answer:- Flow rate is obtained as,

$$Q_0 = \int_0^{\theta_0} \int_0^{2\pi} u_r r^2 \sin \theta d\theta d\phi = 2\pi r^2 \int_0^{\theta_0} u_r \sin \theta d\theta \quad (1)$$

For spherical flow $u_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}$ hence,

$$Q_0 = 2\pi \int_0^{\theta_0} \frac{\partial \psi}{\partial \theta} d\theta = 2\pi (\psi(\theta_0) - \psi(0)). \quad (2)$$

The above shows that $\psi = f(\theta)$ only. Which also shows the flow is purely radial.

- (b) Assume low Reynolds number axisymmetric flow. Convert the coordinates from (r, θ) to (r, η) where $\eta = \cos \theta$. State the governing equation in terms of r and η .

Answer:- For $\eta = \cos \theta$,

$$\frac{\partial}{\partial \theta} = -\sqrt{1-\eta^2} \frac{\partial}{\partial \eta}. \quad (3)$$

Hence,

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{1-\eta^2}{r^2} \frac{\partial^2}{\partial \eta^2}. \quad (4)$$

The governing equation remains $E^2(E^2\psi) = 0$.

- (c) Solve for the streamfunction. Using the appropriate boundary conditions, solve for the radial velocity u_r .

Answer:- Let $\psi = f(\theta) = g(\eta)$. Then,

$$E^2 = \frac{1-\eta^2}{r^2} g'' \quad (5)$$

and

$$\begin{aligned} E^2(E^2\psi) &= \frac{6(1-\eta^2)}{r^4} g'' + \frac{1-\eta^2}{r^4} \frac{\partial}{\partial \eta} ((1-\eta^2)g''' - 2\eta g'') = \\ &= \frac{(1-\eta^2)}{r^4} (6g'' + (1-\eta^2)g'''' - 4\eta g''' - 2g'') = 0 \end{aligned} \quad (6)$$

We obtain the equation

$$4g'' - 4\eta g''' + (1-\eta^2)g'''' = 0. \quad (7)$$

As per given data, consider general solution $g = A\eta^3 + B\eta$. The boundary conditions are $u_r = 0$ at $\theta = \theta_0$. Hence,

$$u_r = \frac{1}{r^2 \sin \theta} \frac{\partial g}{\partial \theta} = -\frac{1}{r^2 \sin \theta} g' \sin \theta = -\frac{g'}{r^2} = -\frac{3A\eta^2 + B}{r^2}. \quad (8)$$

This yields,

$$B = -3A \cos^2 \theta_0. \quad (9)$$

From the flow rate condition,

$$\frac{Q_0}{2\pi} = \int_0^{\theta_0} \frac{\partial \psi}{\partial \theta} d\theta = \psi(\theta_0) - \psi(0) = A (3 \cos^2 \theta_0 - 2 \cos^3 \theta_0 - 1), \quad (10)$$

which yields,

$$A = \frac{Q_0}{2\pi (3 \cos^2 \theta_0 - 2 \cos^3 \theta_0 - 1)}. \quad (11)$$

Computing u_r , we get,

$$u_r = \frac{3Q_0}{2\pi r^2} \frac{\cos^2 \theta - \cos^2 \theta_0}{(2 \cos^3 \theta_0 - 3 \cos^2 \theta_0 + 1)} \quad (12)$$

(d) If the pressure at $r \rightarrow \infty$ is p_∞ , then find the pressure variation inside the cone.

Answer:- Since the flow is purely radial, pressure is constant in θ direction. From the given formula and $\nabla p = \mu \nabla^2 \mathbf{u}$, we get,

$$\frac{\partial p}{\partial r} = \frac{\mu}{r^4 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta 2K \cos \theta \sin \theta) = \frac{2K\mu}{r^4 \sin \theta} (-\sin \theta + 3 \cos^2 \theta \sin \theta), \quad (13)$$

which simplifies to

$$\frac{\partial p}{\partial r} = \frac{2K\mu}{r^4} (3 \cos^2 \theta - 1) \quad (14)$$

where $K = \frac{3Q_0}{2\pi(2 \cos^3 \theta_0 - 3 \cos^2 \theta_0 + 1)}$. Integrate the above in r to obtain p .