

End-semester exam - Solutions

APL361: Advanced Fluid Dynamics (2025-2026-I)

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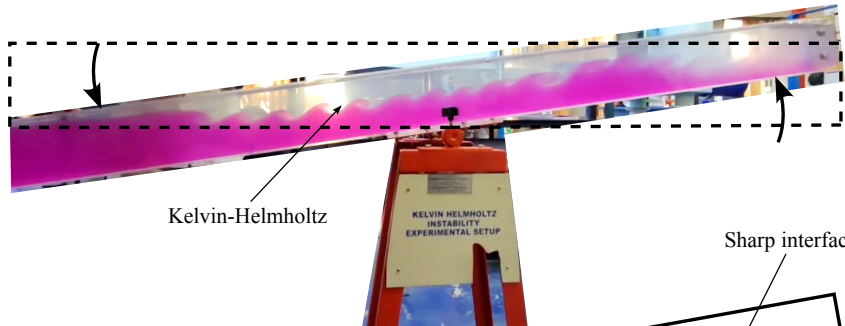
November 22, 2024

Maximum score is $120 + 15$ (Bonus), time limit is 120 minutes. There are two problems in total (on 4 pages). Clearly write all your assumptions and steps. No marks will be given for correct answers without any steps.

Problems

1. Consider the image shown below of a traditional Kelvin-Helmholtz instability setup. Distilled water is filled in a channel (density ρ_1) followed by salt+water (brine) solution with Potassium Permanganate as a dyeing agent (density $\rho_2 = \rho_1 + \Delta\rho$) from the bottom. Both the fluids have viscosity μ . The channel is slowly aligned horizontally so that the heavier fluid (density ρ_2) rests below lighter fluid (density ρ_1) stably. Then, the channel is tilted by an angle θ suddenly at $t = 0$. The heavier fluid rushes to the bottom due to gravity pushing the lighter fluid at the top. This creates a shear layer which develops the Kelvin-Helmholtz instability. Assume uni-directional flow and consider the flow is 2D (in x and y only). The channel is fully closed, from both the ends in x direction as well as in y direction. The flow can be modeled as homogeneous in x .

(a)



(b)

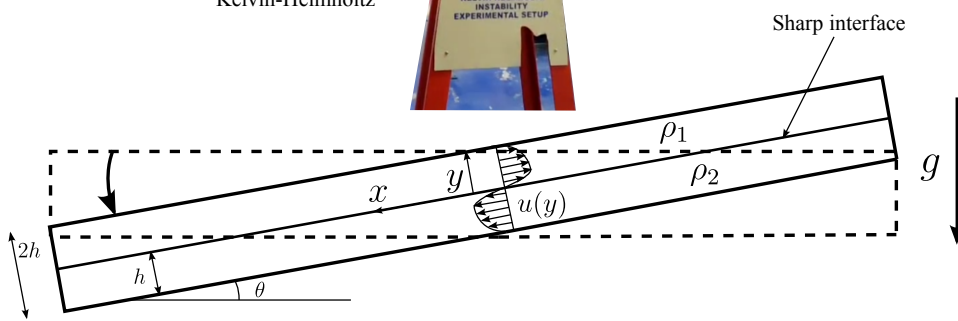


Figure 1: (a) Kelvin-Helmholtz experimental setup in water in the Department of Mechanical Engineering, IIT Delhi (courtesy of Prof. Supreet Singh Bahga). (b) Simplified sketch of the problem.

- (a) State the simplified continuity and momentum equations for both regions 1 and 2. (*Hint :- note that pressure develops due to flow by gravity being obstructed by channel walls*). **(6)**

Answer:- Given uni-directional, homogeneous in x flow, the simplified equations are,

$$\rho_i \frac{\partial u_i}{\partial t} = -\frac{\partial p_i}{\partial x} + \mu \frac{\partial^2 u_i}{\partial y^2} + \rho_i g \sin \theta, \quad (1)$$

$$\frac{\partial p_i}{\partial y} = -\rho_i g \cos \theta, \quad (2)$$

where $i = 1, 2$ for the respective regions.

- (b) Clearly state all the boundary conditions, initial conditions, integral conditions, and interface conditions. There is no surface tension between the two miscible fluids (*Hint :- Carefully observe the qualitative flow velocity profile in the above sketch*). **(6)**

Answer:- At the interface $y = 0$, the kinematic and dynamic interface conditions give

$$u_1(y = 0) = u_2(y = 0), \quad p_1(y = 0, x) = p_2(y = 0, x), \quad \mu \frac{\partial u_1}{\partial y} \Big|_{y=0} = \mu \frac{\partial u_2}{\partial y} \Big|_{y=0}. \quad (3)$$

At the walls, we get

$$u_1(y = h) = 0, \quad u_2(y = -h) = 0. \quad (4)$$

Since the walls of the channel are closed in x , and the flow is uniform in x , there is no net flow-rate in the channel, i.e.,

$$Q = \int_0^h u_1 dy + \int_{-h}^0 u_2 dy = 0. \quad (5)$$

- (c) After some time, steady flow develops in the system. Solve for uni-directional steady flow velocities in both regions 1 and 2. **(20)**

Answer:- At steady state, the x momentum equation yields,

$$-\frac{\partial p_i}{\partial x} + \mu \frac{\partial^2 u_i}{\partial y^2} + \rho_i g \sin \theta = 0. \quad (6)$$

Since $\frac{\partial p_i}{\partial y} = \text{const.}$, pressure can be written as $p_i = p_0 + f_i(x) + g_i(y)$. From momentum equation, we note that $\frac{df_i}{dx} = \text{const.}$ and

$$\mu \frac{d^2 u_i}{dy^2} = \frac{df_i}{dx} - \rho_i g \sin \theta \implies u_i = \left(\frac{df_i}{dx} - \rho_i g \sin \theta \right) \frac{y^2}{2\mu} + C_i y + D_i. \quad (7)$$

From kinematic interface condition, we obtain $D_1 = D_2$, say $= D$. From dynamic interface condition, we obtain

$$p_0 + f_1(x) + g_1(0) = p_0 + f_2(x) + g_2(0), \quad (8)$$

and

$$\frac{du_1}{dy} = \frac{du_2}{dy} \text{ at } y = 0 \implies C_1 = C_2 = C. \quad (9)$$

From y momentum equation, we see that $g_i(y) = -\rho_i g \cos \theta y \implies g_i(y = 0) = 0$. Hence $f_1(x) = f_2(x) = f(x)$. The velocities are given by

$$u_i = \left(\frac{df}{dx} - \rho_i g \sin \theta \right) \frac{y^2}{2\mu} + Cy + D. \quad (10)$$

From the no-slip conditions, we obtain,

$$\left(\frac{df}{dx} - \rho_1 g \sin \theta \right) \frac{h^2}{2\mu} + Ch + D = 0, \quad (11)$$

$$\left(\frac{df}{dx} - \rho_2 g \sin \theta \right) \frac{h^2}{2\mu} - Ch + D = 0. \quad (12)$$

From the zero flow-rate condition, we obtain the additional equation for df/dx , C , and D as

$$\left(\frac{df}{dx} - \rho_1 g \sin \theta \right) \frac{h^3}{6\mu} + \frac{Ch^2}{2} + Dh + \left(\frac{df}{dx} - \rho_2 g \sin \theta \right) \frac{h^3}{6\mu} - \frac{Ch^2}{2} + Dh = 0 \quad (13)$$

which yields,

$$D = \left(\frac{(\rho_1 + \rho_2)}{2} g \sin \theta - \frac{df}{dx} \right) \frac{h^2}{6\mu}. \quad (14)$$

From the above two equations, we obtain,

$$D = \left(\frac{(\rho_1 + \rho_2)}{2} g \sin \theta - \frac{df}{dx} \right) \frac{h^2}{2\mu}. \quad (15)$$

The only possible solution is then $D = 0$ and $\frac{df}{dx} = \frac{\rho_1 + \rho_2}{2} g \sin \theta$. This yields

$$C = \frac{\rho_1 - \rho_2}{2\mu} gh \sin \theta. \quad (16)$$

and the velocity in regions 1 and 2 as,

$$u_{1,2} = \left(\frac{\rho_{2,1} - \rho_{1,2}}{2} g \sin \theta \right) \frac{y^2}{2\mu} + \frac{\rho_1 - \rho_2}{2\mu} ghy \sin \theta. \quad (17)$$

(d) Solve for pressure in regions 1 and 2. **(3)**

Answer:- The pressure is obtained from df/dx and g_i as ,

$$p_i = p_0 + \left(\frac{\rho_1 + \rho_2}{2} g \sin \theta \right) x - \rho_i g \cos \theta y, \quad (18)$$

where p_0 is some arbitrary pressure value.

(e) **Kelvin-Helmholtz instability between miscible inviscid flow streams**

- i. Now to model the Kelvin-Helmholtz instability between the miscible flow streams, assume the flows are *inviscid* with speeds $\int_0^h u dy = hU_1$ and $\int_{-h}^0 u dy = hU_2$, where $u(y)$ is the solution obtained above. Clearly derive the expressions of U_1 and U_2 . State the governing equations in region 1 and 2 and the interface conditions. Ignore the pressures evaluated in part 1d and find new relation of pressures in both the regions. **(2)**

Answer:- From flow rate condition above $U_1 = -U_2$. So we have to evaluate only one of them. Integrating $u_1(y)$ we get,

$$\int_0^h u_1 dy = U_1 h = \frac{(\rho_2 - \rho_1) g \sin \theta h^3}{12\mu} + \frac{(\rho_1 - \rho_2) g \sin \theta h^3}{4\mu} = -\Delta \rho g \sin \theta \frac{h^3}{6\mu}. \quad (19)$$

The above confirms that the flow of fluid 1 is in $-x$ direction. Assuming inviscid flow, the governing equations are simply,

$$\nabla^2 \phi_i = 0, \quad (20)$$

where ϕ_1 and ϕ_2 are velocity potentials in regions 1 and 2. Pressure is related through the Bernoulli's equation

$$\frac{\partial \phi_i}{\partial t} + \frac{|\nabla \phi_i|^2}{2} + \frac{p_i}{\rho_i} + \rho_i g \cos \theta y = C_i(t). \quad (21)$$

Since the flow is inviscid (and in x direction only), the pressures we obtain are

$$p_i = p_0 - \rho_i g \cos \theta y. \quad (22)$$

Since there is no surface tension $p_1 = p_2$ at $y = 0$ for all x . This can only be satisfied if p_1 and p_2 have identical variation in x . If we take previous part's answer of $\frac{\rho_1 + \rho_2}{2} g x \sin \theta$ then forces will not cancel and the inviscid flow will accelerate. Hence, we can ignore any gravity in x direction for inviscid flow modeling and assume pressure uniform in x .

- ii. Assume small perturbations in the sharp interface. Linearize the governing equations and the interface conditions. **(5)**

Answer:- Assuming perturbation in interface as $\xi'(x, t)$, flow potentials as $\phi'_i(x, y, t)$ and pressure perturbations $p'_i(x, y, t)$, we obtain the following. From inviscid flow assumption,

$$\nabla^2 \phi'_i = 0. \quad (23)$$

From dynamic interface condition,

$$p'_1 = p'_2 \text{ at } y = \xi'(x, t). \quad (24)$$

After linearizing (as done in class), we get $p'_1 = p'_2$ at $y = 0$. From kinematic interface condition,

$$\frac{\partial \phi'_i}{\partial y} = \frac{\partial \xi'}{\partial t} + U_i \frac{\partial \xi'}{\partial x} \text{ at } y = \xi'(x, t). \quad (25)$$

After linearizing, we can apply this at $y = 0$. From dynamic interface condition, using the Bernoulli's equation, we obtain,

$$\rho_1 \left(\frac{\partial \phi'_1}{\partial t} + U_1 \frac{\partial \phi'_1}{\partial x} + g \cos \theta \xi' \right) = \rho_2 \left(\frac{\partial \phi'_2}{\partial t} + U_2 \frac{\partial \phi'_2}{\partial x} + g \cos \theta \xi' \right) \text{ at } y = 0. \quad (26)$$

- iii. Clearly state the required normal modes assumption for all variables. **(3)**

Answer:- The required normal modes assumptions for all variables are,

$$\phi'_i = \int \int \hat{\phi}_i(y, k, \omega) e^{i(kx - \omega t)} dk d\omega, \quad (27)$$

$$\xi' = \int \int \hat{\xi}(k, \omega) e^{i(kx - \omega t)} dk d\omega. \quad (28)$$

- iv. Substitute the normal modes assumption and derive the limits of stability of the flow. **(10)**

Answer:- From the inviscid flow equations, we obtain,

$$\nabla^2 \phi'_i = 0 \implies \frac{d^2 \hat{\phi}}{dy^2} - k^2 \hat{\phi} = 0 \implies \hat{\phi}_i = A_i e^{ky} + B_i e^{-ky}. \quad (29)$$

The boundaries at $y = \pm h$ are hard walls (closed channel). Hence $v = 0$ at $y = \pm h$.

From the potential ($\frac{d\hat{\phi}}{dy} = 0$), we obtain

$$B_1 = A_1 e^{2kh}, \quad B_2 = A_2 e^{-2kh}. \quad (30)$$

Further substituting in the dynamic condition, we obtain,

$$\rho_1 \left(g \cos \theta \hat{\xi} + i(U_1 k - \omega)(A_1 + B_1) \right) = \rho_2 \left(g \cos \theta \hat{\xi} + i(U_2 k - \omega)(A_2 + B_2) \right). \quad (31)$$

From kinematic interface conditions, we obtain,

$$k(A_1 - B_1) = i(kU_1 - \omega)\hat{\xi}, \quad k(A_2 - B_2) = i(kU_2 - \omega)\hat{\xi} \quad (32)$$

Substituting in the dynamic condition, we obtain,

$$\omega^2 - 2\omega k \left(\frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \right) + \frac{\rho_1 U_1^2 k^2 + \rho_2 U_2^2 k^2}{\rho_1 + \rho_2} - \frac{k(\rho_1 - \rho_2)g \cos \theta}{(\rho_1 + \rho_2)f} = 0, \quad (33)$$

where $f = \frac{1 + e^{2kh}}{1 - e^{2kh}}$. The solutions are

$$\omega = \frac{(\rho_1 U_1 + \rho_2 U_2)k}{\rho_1 + \rho_2} \pm \sqrt{-\frac{\rho_1 \rho_2 (U_1 - U_2)^2}{(\rho_1 + \rho_2)^2} + \frac{(\rho_1 - \rho_2)kg \cos \theta}{(\rho_1 + \rho_2)f}} \quad (34)$$

- Full points if even quadratic was derived correctly accounting for finite heights in y .
- Full points if finite heights ignored in y and the full solution of the quadratic was derived.

(f) Kelvin-Helmholtz instability between immiscible inviscid flow streams

- i. Assume the same U_1 and U_2 as in part **1(e)i** but now consider the two fluids to be immiscible with a surface tension coefficient between the two layers as σ . State the set of full governing equations and interface conditions for this case. Assume inviscid flow. **(2)**

Answer:- The only change for immiscible fluids with surface tension coefficient is in the dynamic interface condition which yields,

$$\left. \underline{T} \cdot \hat{n} \right|_2 - \left. \underline{T} \cdot \hat{n} \right|_1 = 2\mathcal{H}\sigma \hat{n} = -\sigma \nabla \cdot \hat{n} \quad (35)$$

as derived in class (and class notes). Since the flow is assumed inviscid, only pressure is the part of stress-tensor (recall $T_{ij} = -p\delta_{ij}$) which yields,

$$p_1 - p_2 = -\sigma \nabla \cdot \hat{n}. \quad (36)$$

- ii. Assume small perturbations in the sharp interface. Linearize the governing equations and the interface conditions. **(10)**

Answer:- The governing equations and kinematic interface conditions remain same as the above part, i.e.,

$$\nabla^2 \phi'_i = 0, \quad (37)$$

and

$$\frac{\partial \phi'_i}{\partial y} = \frac{\partial \xi'}{\partial t} + U_i \frac{\partial \xi'}{\partial x} \text{ at } y = \xi'(x, t). \quad (38)$$

For linearizing the dynamic interface condition, we first evaluate $\nabla \cdot \hat{n}$ as ,

$$\hat{n} = \frac{\nabla(y - \epsilon \xi')}{|\nabla(y - \epsilon \xi')|} = \frac{\hat{j} - \epsilon \frac{\partial \xi'}{\partial x} \hat{i}}{\sqrt{1 + \epsilon^2 \frac{\partial \xi'}{\partial x}^2}} \implies \nabla \cdot \hat{n} = -\epsilon \frac{\partial^2 \xi'}{\partial x^2} + \mathcal{O}(\epsilon^2). \quad (39)$$

Since the base interface is undeformed, the pressure perturbations are related as,

$$p'_1 - p'_2 = \sigma \frac{\partial^2 \xi'}{\partial x^2}. \quad (40)$$

- iii. Substitute the normal modes assumption and derive the limits of stability of the flow. **(10)**

Answer:- Substituting the identical normal-modes assumption as the previous part, we obtain

$$\nabla^2 \phi'_i = 0 \implies \frac{d^2 \hat{\phi}}{dy^2} - k^2 \hat{\phi} = 0 \implies \hat{\phi}_i = A_i e^{ky} + B_i e^{-ky}. \quad (41)$$

with

$$B_1 = A_1 e^{2kh}, \quad B_2 = A_2 e^{-2kh}. \quad (42)$$

Further substituting in the dynamic condition, we obtain,

$$\rho_2 \left(g \cos \theta \hat{\xi} + i(U_2 k - \omega)(A_2 + B_2) \right) - \rho_1 \left(g \cos \theta \hat{\xi} + i(U_1 k - \omega)(A_1 + B_1) \right) = -k^2 \sigma. \quad (43)$$

From kinematic interface conditions, we obtain,

$$k(A_1 - B_1) = i(kU_1 - \omega)\hat{\xi}, \quad k(A_2 - B_2) = i(kU_2 - \omega)\hat{\xi} \quad (44)$$

Substituting in the dynamic condition, we obtain,

$$\omega^2 - 2\omega k \left(\frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \right) + \frac{\rho_1 U_1^2 k^2 + \rho_2 U_2^2 k^2}{\rho_1 + \rho_2} - \frac{k(\rho_1 - \rho_2)g \cos \theta}{(\rho_1 + \rho_2)f} + \frac{k^3 \sigma}{(\rho_1 + \rho_2)f} = 0, \quad (45)$$

where $f = \frac{1 + e^{2kh}}{1 - e^{2kh}}$. The solutions are

$$\omega = \frac{(\rho_1 U_1 + \rho_2 U_2)k}{\rho_1 + \rho_2} \pm \sqrt{-\frac{\rho_1 \rho_2 (U_1 - U_2)^2}{(\rho_1 + \rho_2)^2} + \frac{(\rho_1 - \rho_2)kg \cos \theta - \sigma k^3}{(\rho_1 + \rho_2)f}} \quad (46)$$

Note that $\rho_1 - \rho_2 = -\Delta\rho$. Also, for very large h , $f < 0$.

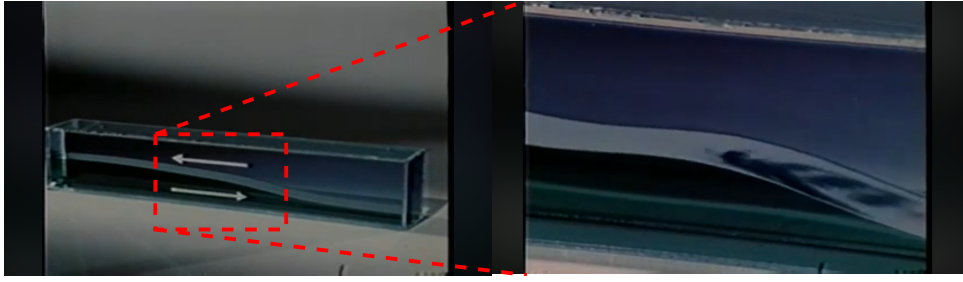


Figure 2: Kelvin-Helmholtz setup with immiscible fluids.

- iv. In the figure shown below, with two immiscible fluids, a strong shear layer is established in the channel. However, the Kelvin-Helmholtz instability is only weakly visible. Compare the results from part 1(e)iv and 1(f)iii and comment on this observation (only first two sentences will be considered for grading). **(3)**

Answer:- Substituting for $f = -|f|$ in the dispersion relation we obtain,

$$\omega = \frac{(\rho_1 U_1 + \rho_2 U_2)k}{\rho_1 + \rho_2} \pm \sqrt{-\frac{\rho_1 \rho_2 (U_1 - U_2)^2}{(\rho_1 + \rho_2)^2} + \frac{\Delta \rho k g \cos \theta + \sigma k^3}{(\rho_1 + \rho_2)|f|}} \quad (47)$$

For no density differences and no surface tension, the situation is always unstable (standard Kelvin-Helmholtz without density differences). With density differences, we the frequency becomes complex only when

$$\frac{\rho_1 \rho_2 (U_1 - U_2)^2}{(\rho_1 + \rho_2)^2} > \frac{\Delta \rho k g \cos \theta + \sigma k^3}{(\rho_1 + \rho_2)|f|}. \quad (48)$$

Hence, high density differences require stronger shear $|U_1 - U_2|$ for Kelvin-Helmholtz instability. Surface tension imposes further stabilizing forces. Due to this stabilization, the waves visible are very less prominent in the immiscible example.

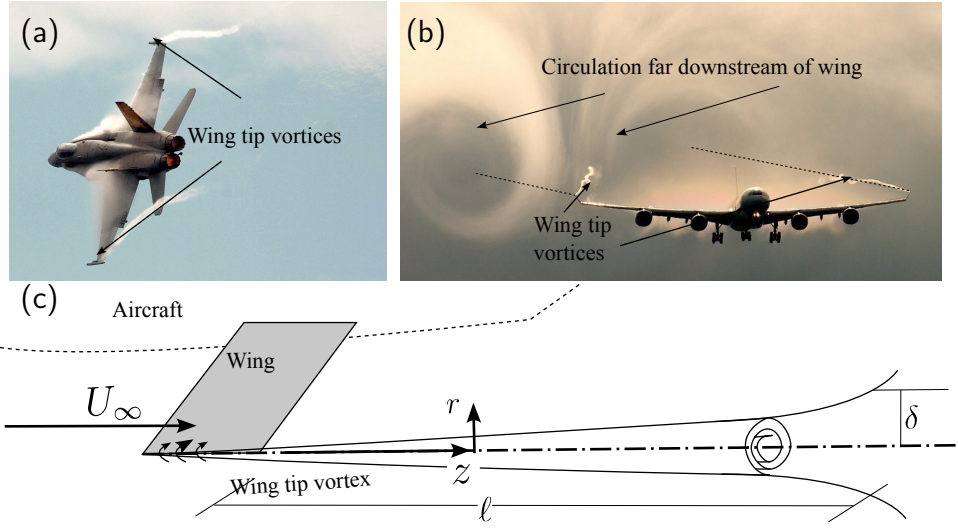


Figure 3: (a) Wing tip vortices near wings of an aircraft (b) Far field of wing tip vortices (c) Wing tip vortex modeling in cylindrical coordinates.

2. Wing tip vortices form when a finite wing passes through air as shown. Here we will consider simplified model of an axisymmetric wing tip vortex induced by a wing in a uniform flow of speed U_∞ . The wing induces a rotating vortex (azimuthal velocity in $\hat{e}_\theta$) direction which gets convected downstream as shown in the figure. Assume axis-symmetric incompressible flow of air with density ρ and kinematic viscosity ν .

- (a) Write the continuity and all momentum equations for the axisymmetric flow in the wing tip vortex. Take non-zero azimuthal velocity u_θ . **(5)**

Answer:- In the axisymmetric steady flow with u_θ , variation of both velocity and pressure in θ direction can be ignored which yields the equations,

$$\frac{\partial u_z}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} (r u_r) = 0, \quad (49)$$

$$u_z \frac{\partial u_z}{\partial z} + u_r \frac{\partial u_z}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \nabla^2 u_z, \quad (50)$$

$$u_z \frac{\partial u_r}{\partial z} + u_r \frac{\partial u_r}{\partial r} - \frac{u_\theta^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left(\nabla^2 u_r - \frac{u_r}{r^2} \right), \quad (51)$$

$$u_z \frac{\partial u_\theta}{\partial z} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_r u_\theta}{r} = \nu \left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} \right), \quad (52)$$

where $\nabla^2 = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}$.

- (b) Using the following scales (choose an appropriate scale for pressure), non-dimensionalize the governing equations and evaluate the appropriate scale for the radial velocity.

$$[u_z] = U_\infty, [u_\theta] = U_\infty, [z] = \ell, [r] = \delta.$$

Use $()^*$ for dimensionless variables. **(5)**

Answer:- For variation of u_z in z , the appropriate scale of u_r is obtained from continuity as $[u_r] = \frac{U_\infty \delta}{\ell}$. Choosing pressure scale as $p_0 = \rho U_\infty^2$, after non-dimensionalization, we

obtain,

$$\frac{\partial u_z^*}{\partial z^*} + \frac{1}{r^*} \frac{\partial}{\partial r^*} (r^* u_r^*) = 0, \quad (53)$$

$$\frac{U_\infty^2}{\ell} \left(u_z^* \frac{\partial u_z^*}{\partial z^*} + u_r^* \frac{\partial u_z^*}{\partial r^*} \right) = -\frac{U_\infty^2}{\ell} \frac{\partial p^*}{\partial z^*} + \frac{\nu U_\infty}{\delta^2} \left(\frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial u_z^*}{\partial r^*} \right) + \frac{\delta^2}{\ell^2} \frac{\partial^2 u_z^*}{\partial z^{*2}} \right), \quad (54)$$

$$\frac{U_\infty^2 \delta}{\ell^2} \left(u_z^* \frac{\partial u_r^*}{\partial z^*} + u_r^* \frac{\partial u_r^*}{\partial r^*} - \frac{\ell^2}{\delta^2} \frac{u_\theta^{*2}}{r^*} \right) = -\frac{U_\infty^2}{\delta} \frac{\partial p^*}{\partial r^*} + \frac{\nu U_\infty}{\ell \delta} \left(\frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial u_r^*}{\partial r^*} \right) + \frac{\delta^2}{\ell^2} \frac{\partial^2 u_r^*}{\partial z^{*2}} - \frac{u_r^*}{r^{*2}} \right), \quad (55)$$

$$\frac{U_\infty^2}{\ell} \left(u_z^* \frac{\partial u_\theta^*}{\partial z^*} + u_r^* \frac{\partial u_\theta^*}{\partial r^*} + \frac{u_r^* u_\theta^*}{r^*} \right) = \frac{\nu U_\infty}{\delta^2} \left(\frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial u_\theta^*}{\partial r^*} \right) + \frac{\delta^2}{\ell^2} \frac{\partial^2 u_\theta^*}{\partial z^{*2}} - \frac{u_\theta^*}{r^{*2}} \right) \quad (56)$$

- (c) Under the conditions $\frac{\delta}{\ell} = \epsilon \ll 1$ and $\frac{\nu \ell}{U_\infty \delta^2} = 1$, derive the leading order dimensionless equations. Feel free to drop $()^*$ from this part onwards. **(5)**

Answer:- The continuity equation remains as is. The momentum equations simplify since $\frac{\nu \ell}{U_\infty \delta^2} = \frac{1}{Re_\ell \epsilon^2} = 1$. At leading order, we obtain,

$$u_z^* \frac{\partial u_z^*}{\partial z^*} + u_r^* \frac{\partial u_z^*}{\partial r^*} = -\frac{\partial p^*}{\partial z^*} + \frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial u_z^*}{\partial r^*} \right), \quad (57)$$

$$\frac{\partial p^*}{\partial r^*} - \frac{u_\theta^{*2}}{r^*} = 0, \quad (58)$$

$$u_z^* \frac{\partial u_\theta^*}{\partial z^*} + u_r^* \frac{\partial u_\theta^*}{\partial r^*} + \frac{u_r^* u_\theta^*}{r^*} = \left(\frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial u_\theta^*}{\partial r^*} \right) - \frac{u_\theta^*}{r^{*2}} \right). \quad (59)$$

- (d) Without the wing, the flow dimensional velocity is $\underline{u} = U_\infty \hat{e}_z$ and $p = p_\infty$. Consider that the wing tip vortex as a perturbation such that the dimensionless velocity field can be decomposed as,

$$\underline{u}^* = \hat{e}_z + \epsilon(u(r, z)\hat{e}_z + v(r, z)\hat{e}_r + w(r, z)\hat{e}_\theta),$$

where $\epsilon \ll 1$. Substitute this decomposition in solution of 2c and obtain a further leading order set of equations in ϵ . (*Hint-This is same as the linearization procedure of stability analysis*) **(5)**

Answer:- After linearization, we obtain (dropping the $()^*$ notation from dimensionless variables),

$$\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right), \quad (60)$$

$$\frac{\partial p}{\partial r} = \frac{w^2}{r}, \quad (61)$$

$$\frac{\partial w}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial w}{\partial r} \right) - \frac{w}{r^2} \quad (62)$$

- (e) Solve the equation for $w(r, z)$ using the self-similarity guess as $w(r, z) = \frac{f(\eta)}{r}$ where $\eta = \frac{r^2}{z}$. Use the condition of finite $w(r, z)$ as $r \rightarrow 0$. Comment upon the remaining arbitrary constant **(15)**

Answer:- Substituting $w = \frac{f}{r}$, we obtain,

$$4f'' + f' = 0 \implies f = Ce^{-\eta/4} + D. \quad (63)$$

For $\eta = 0$, and $r \rightarrow 0$, $w(r, z)$ will be finite if $f(\eta = 0) = 0$ which yields,

$$f = C (1 - e^{-\eta/4}). \quad (64)$$

For $z \rightarrow 0$, $\eta \rightarrow \infty$ and $f \rightarrow C$. Around a circular path in the $r - \theta$ plane, circulation will be equal to $2\pi r w = 2\pi f$. Hence, C is $1/2\pi$ times the initial circulation induced by the wing.

- (f) Solve for pressure. Assume that all the variation of pressure is due to variation of $w(r, z)$ only. Leave the integration of the form $\int \frac{e^{-\eta}}{\eta} d\eta$ as is. Qualitatively sketch the variation of pressure. Start by finding pressure (or its derivative) for small η and large η . **(5)**

Answer:- For pressure, we use the equation

$$\frac{\partial p}{\partial r} = \frac{w^2}{r} = \frac{f^2}{r^3} = C^2 \frac{(1 - e^{-\eta/4})^2}{r^3} \implies p_\infty - p = C^2 \int_r^\infty \frac{(1 - e^{-\eta/4})^2}{r^3} dr. \quad (65)$$

Changing the integration variable,

$$p_\infty - p = \frac{C^2}{2z} \int_\eta^\infty \frac{(1 - e^{-\eta/4})^2}{\eta^2} d\eta = \frac{C^2}{2z} \left(\frac{(1 - e^{-\eta/4})^2}{\eta} - \frac{1}{2} \int_\eta^\infty \frac{e^{-\eta/4}}{\eta} d\eta + \frac{1}{2} \int_\eta^\infty \frac{e^{-\eta/2}}{\eta} d\eta \right) \quad (66)$$

To get the qualitative sketch of pressure, we note that for small η ,

$$\frac{\partial p}{\partial \eta} = \frac{C^2}{2z} \frac{\left(1 - \left(1 - \frac{\eta}{4} + \frac{\eta^2}{32} - \dots\right)\right)^2}{\eta^2} = \frac{C^2}{2z} \left(\frac{1}{4} - \frac{\eta}{32} + \dots\right)^2 \approx \frac{C^2}{2z} \left(\frac{1}{4} - \frac{\eta}{16}\right). \quad (67)$$

Hence, $p - p_\infty \propto \frac{\eta}{4}$ for small η . For large η ,

$$\frac{\partial p}{\partial \eta} \approx \frac{C^2}{2z} \frac{1}{\eta^2} \implies p - p_\infty \rightarrow -\frac{1}{\eta}. \quad (68)$$

Since pressure has to approach p_∞ for large η , the qualitative sketch of pressure $p - p_\infty$ looks like, The pressure dips inside the wing-tip vortices. Assuming these vortices form

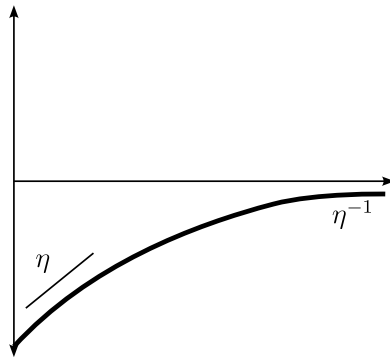


Figure 4: Qualitative variation of pressure.

adiabatically, the temperature also decreases resulting in water-vapour condensation.

- (g) (Bonus) Substitute the above expressions of pressure and w to obtain a linearized governing equation for $u(r, z)$. Integrate this equation over an $x = \text{const.}$ plane. **(15)**

Answer:- Writing pressure as

$$p - p_0 = \frac{C^2}{2z} P(\eta), \quad (69)$$

we can substitute in the u equation to obtain,

$$\frac{\partial u}{\partial z} - \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) = \frac{C^2}{2z^2} (P(\eta) + \eta P'(\eta)). \quad (70)$$

Integrating the equation in plane, we obtain,

$$\frac{d}{dz} \int_0^\infty u 2\pi r dr - 2\pi \left(r \frac{\partial u}{\partial r} \Big|_{r \rightarrow \infty} - r \frac{\partial u}{\partial r} \Big|_{r=0} \right) = \frac{2\pi C^2}{2z^2} \int_0^\infty \frac{d(\eta P)}{d\eta} r dr. \quad (71)$$

Since, far away, gradients vanish, the second term in brackets on LHS vanishes. Changing the integral sign, we obtain,

$$\frac{d}{dz} \int_0^\infty u r dr = \frac{C^2}{4z} \int_0^\infty \frac{d(\eta P)}{d\eta} d\eta = \frac{C^2}{4z} \left((\eta P) \Big|_{\eta \rightarrow 0} - (\eta P) \Big|_{\eta=0} \right). \quad (72)$$

From our previous derivation, we obtain,

$$\frac{d}{dz} \int_0^\infty u r dr = \frac{C^2}{4z} \implies \int_0^\infty u r dr = \frac{C^2}{4} \log z \quad (73)$$

Important relations

- $\nabla f = \frac{\partial f}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{e}_\theta + \frac{\partial f}{\partial z} \hat{e}_z.$
- $\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}.$
- $\nabla \cdot \underline{u} = \frac{1}{r} \frac{\partial(r u_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}.$
- $\nabla \times \underline{u} = \left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) \hat{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \hat{e}_\theta + \left(\frac{1}{r} \frac{\partial(r u_\theta)}{\partial r} - \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{e}_z.$
- $\nabla^2 \underline{u} = \left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right) \hat{e}_r + \left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right) \hat{e}_\theta + \nabla^2 u_z \hat{e}_z.$
- $\underline{u} \cdot \nabla f = u_r \frac{\partial f}{\partial r} + \frac{u_\theta}{r} \frac{\partial f}{\partial \theta} + u_z \frac{\partial f}{\partial z}.$
- $\underline{u} \cdot \nabla \underline{u} = \left(\underline{u} \cdot \nabla u_r - \frac{u_\theta^2}{r} \right) \hat{e}_r + \left(\underline{u} \cdot \nabla u_\theta + \frac{u_\theta u_r}{r} \right) \hat{e}_\theta + \underline{u} \cdot \nabla u_z \hat{e}_z.$
- Navier-Stokes for incompressible flow of a Newtonian fluid: $\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \underline{u}$