

# End-semester exam

## APL361: Advanced Fluid Dynamics (2025-2026-I)

Department of Applied Mechanics, IIT Delhi

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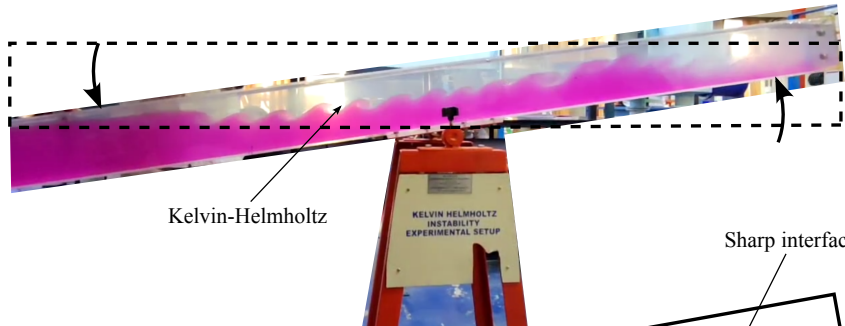
November 22, 2024

Maximum score is 120 + 15 (Bonus), time limit is 120 minutes. There are two problems in total (on 4 pages). Clearly write all your assumptions and steps. No marks will be given for correct answers without any steps.

### Problems

1. Consider the image shown below of a traditional Kelvin-Helmholtz instability setup. Distilled water is filled in a channel (density  $\rho_1$ ) followed by salt+water (brine) solution with Potassium Permanganate as a dyeing agent (density  $\rho_2 = \rho_1 + \Delta\rho$ ) from the bottom. Both the fluids have viscosity  $\mu$ . The channel is slowly aligned horizontally so that the heavier fluid (density  $\rho_2$ ) rests below lighter fluid (density  $\rho_1$ ) stably. Then, the channel is tilted by an angle  $\theta$  suddenly at  $t = 0$ . The heavier fluid rushes to the bottom due to gravity pushing the lighter fluid at the top. This creates a shear layer which develops the Kelvin-Helmholtz instability. Assume uni-directional flow and consider the flow is 2D (in  $x$  and  $y$  only). The channel is fully closed, from both the ends in  $x$  direction as well as in  $y$  direction. The flow can be modeled as homogeneous in  $x$ .

(a)



(b)

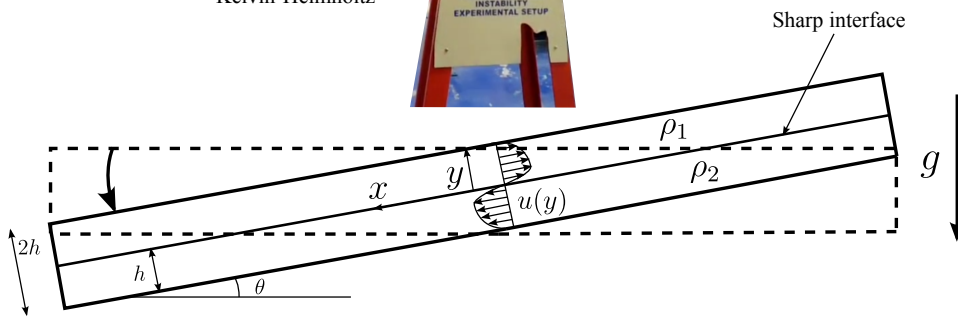


Figure 1: (a) Kelvin-Helmholtz experimental setup in water in the Department of Mechanical Engineering, IIT Delhi (courtesy of Prof. Supreet Singh Bahga). (b) Simplified sketch of the problem.

- (a) State the simplified continuity and momentum equations for both regions 1 and 2. (*Hint :- note that pressure develops due to flow by gravity being obstructed by channel walls*). **(6)**
- (b) Clearly state all the boundary conditions, initial conditions, integral conditions, and interface conditions. There is no surface tension between the two miscible fluids (*Hint :- Carefully observe the qualitative flow velocity profile in the above sketch*). **(6)**

(c) After some time, steady flow develops in the system. Solve for uni-directional steady flow velocities in both regions 1 and 2. **(20)**

(d) Solve for pressure in regions 1 and 2. **(3)**

**(e) Kelvin-Helmholtz instability between miscible inviscid flow streams**

- i. Now to model the Kelvin-Helmholtz instability between the miscible flow streams, assume the flows are *inviscid* with speeds  $\int_0^h u dy = hU_1$  and  $\int_{-h}^0 u dy = hU_2$ , where  $u(y)$  is the solution obtained above. Clearly derive the expressions of  $U_1$  and  $U_2$ . State the governing equations in region 1 and 2 and the interface conditions. Ignore the pressures evaluated in part 1d and find new relation of pressures in both the regions. **(2)**
- ii. Assume small perturbations in the sharp interface. Linearize the governing equations and the interface conditions. **(5)**
- iii. Clearly state the required normal modes assumption for all variables. **(3)**
- iv. Substitute the normal modes assumption and derive the limits of stability of the flow. **(10)**

**(f) Kelvin-Helmholtz instability between immiscible inviscid flow streams**

- i. Assume the same  $U_1$  and  $U_2$  as in part 1(e)i but now consider the two fluids to be immiscible with a surface tension coefficient between the two layers as  $\sigma$ . State the set of full governing equations and interface conditions for this case. Assume inviscid flow. **(2)**
- ii. Assume small perturbations in the sharp interface. Linearize the governing equations and the interface conditions. **(10)**
- iii. Substitute the normal modes assumption and derive the limits of stability of the flow. **(10)**
- iv. In the figure shown below, with two immiscible fluids, a strong shear layer is established in the channel. However, the Kelvin-Helmholtz instability is only weakly visible. Compare the results from part 1(e)iv and 1(f)iii and comment on this observation (only first two sentences will be considered for grading). **(3)**

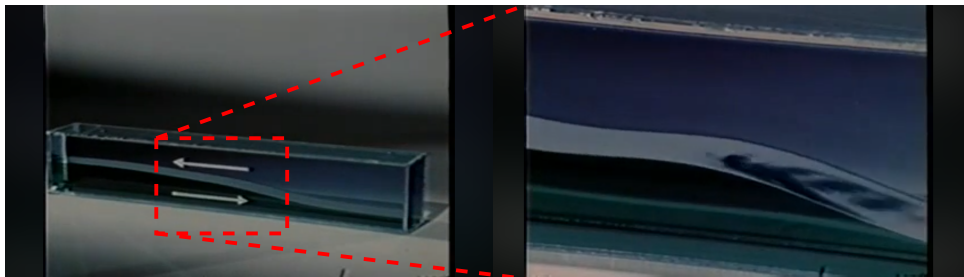


Figure 2: Kelvin-Helmholtz setup with immiscible fluids.

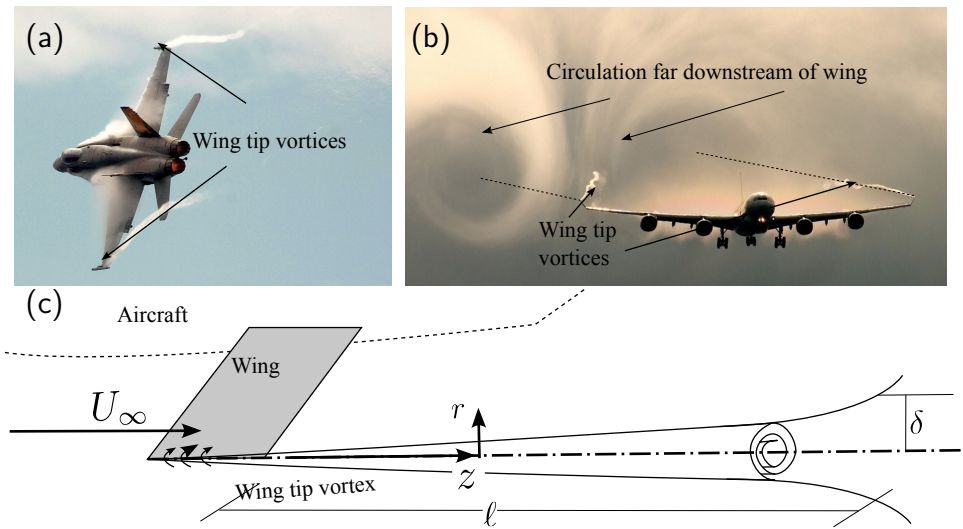


Figure 3: (a) Wing tip vortices near wings of an aircraft (b) Far field of wing tip vortices (c) Wing tip vortex modeling in cylindrical coordinates.

2. Wing tip vortices form when a finite wing passes through air as shown. Here we will consider simplified model of an axisymmetric wing tip vortex induced by a wing in a uniform flow of speed  $U_\infty$ . The wing induces a rotating vortex (azimuthal velocity in  $\hat{e}_\theta$ ) direction which gets convected downstream as shown in the figure. Assume axis-symmetric incompressible flow of air with density  $\rho$  and kinematic viscosity  $\nu$ .

- (a) Write the continuity and all momentum equations for the axisymmetric flow in the wing tip vortex. Take non-zero azimuthal velocity  $u_\theta$ . **(5)**
- (b) Using the following scales (choose an appropriate scale for pressure), non-dimensionalize the governing equations and evaluate the appropriate scale for the radial velocity.

$$[u_z] = U_\infty, [u_\theta] = U_\infty, [z] = \ell, [r] = \delta.$$

Use  $()^*$  for dimensionless variables. **(5)**

- (c) Under the conditions  $\frac{\delta}{\ell} = \epsilon \ll 1$  and  $\frac{\nu \ell}{U_\infty \delta^2} = 1$ , derive the leading order dimensionless equations. Feel free to drop  $()^*$  from this part onwards. **(5)**
- (d) Without the wing, the flow dimensional velocity is  $\underline{u} = U_\infty \hat{e}_z$  and  $p = p_\infty$ . Consider that the wing tip vortex as a perturbation such that the dimensionless velocity field can be decomposed as,

$$\underline{u}^* = \hat{e}_z + \epsilon(u(r, z)\hat{e}_z + v(r, z)\hat{e}_r + w(r, z)\hat{e}_\theta),$$

where  $\epsilon \ll 1$ . Substitute this decomposition in solution of 2c and obtain a further leading order set of equations in  $\epsilon$ . (*Hint-This is same as the linearization procedure of stability analysis*) **(5)**

- (e) Solve the equation for  $w(r, z)$  using the self-similarity guess as  $w(r, z) = \frac{f(\eta)}{r}$  where  $\eta = \frac{r^2}{z}$ . Use the condition of finite  $w(r, z)$  as  $r \rightarrow 0$ . Comment upon the remaining arbitrary constant **(15)**
- (f) Solve for pressure. Assume that all the variation of pressure is due to variation of  $w(r, z)$

only. Leave the integration of the form  $\int \frac{e^{-\eta}}{\eta} d\eta$  as is. Qualitatively sketch the variation of pressure. Start by finding pressure (or its derivative) for small  $\eta$  and large  $\eta$ . **(5)**

(g) (Bonus) Substitute the above expressions of pressure and  $w$  to obtain a linearized governing equation for  $u(r, z)$ . Integrate this equation over an  $x = \text{const.}$  plane. **(15)**

## Important relations

- $\nabla f = \frac{\partial f}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{e}_\theta + \frac{\partial f}{\partial z} \hat{e}_z.$
- $\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}.$
- $\nabla \cdot \underline{u} = \frac{1}{r} \frac{\partial(r u_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}.$
- $\nabla \times \underline{u} = \left( \frac{1}{r} \frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right) \hat{e}_r + \left( \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right) \hat{e}_\theta + \left( \frac{1}{r} \frac{\partial(r u_\theta)}{\partial r} - \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \hat{e}_z.$
- $\nabla^2 \underline{u} = \left( \nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right) \hat{e}_r + \left( \nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right) \hat{e}_\theta + \nabla^2 u_z \hat{e}_z.$
- $\underline{u} \cdot \nabla f = u_r \frac{\partial f}{\partial r} + \frac{u_\theta}{r} \frac{\partial f}{\partial \theta} + u_z \frac{\partial f}{\partial z}.$
- $\underline{u} \cdot \nabla \underline{u} = \left( \underline{u} \cdot \nabla u_r - \frac{u_\theta^2}{r} \right) \hat{e}_r + \left( \underline{u} \cdot \nabla u_\theta + \frac{u_\theta u_r}{r} \right) \hat{e}_\theta + \underline{u} \cdot \nabla u_z \hat{e}_z.$
- Navier-Stokes for incompressible flow of a Newtonian fluid:  $\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \underline{u}$