

APL361 (2024-2025-I): Mid-term exam solutions

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Maximum score is 120, time limit is 120 minutes. There are three problems in total. Clearly write all your assumptions and steps. No marks will be given for correct answers without any steps. **Answer:- I have skipped steps in these solutions for brevity. Not all solutions (in terms of completeness of steps) here are representative of the solutions to be written by a student in an exam.**

Important relations

- Axisymmetric incompressible flow in spherical coordinates:

$$1. u_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}, u_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r},$$

$$2. E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right)$$

$$3. \nabla^2 \mathbf{u} = \left(\frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u_r}{\partial \theta} \right) - \frac{2}{r^2 \sin \theta} \frac{\partial (u_\theta \sin \theta)}{\partial \theta} \right) \hat{e}_r + \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (u_\theta \sin \theta) \right) + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right) \hat{e}_\theta$$

- One root of $x^3 - 6x^2 - x + 30 = 0$ is $x = 3$.

Problems

1. In class, we have solved for velocity which yields a prescribed vorticity field. A velocity dilatational field is defined as $\nabla \cdot \underline{u}$. Using the Helmholtz' decomposition theorem and Green's function in 3D, derive the velocity field which yields a prescribed dilatational field $D(\underline{x})$. That is, solve for $\underline{u}$ such that $\nabla \cdot \underline{u} = D(\underline{x})$. **(20)**

Answer:- From Helmholtz' decomposition,

$$\underline{u} = \nabla \phi + \nabla \times \underline{A}. \quad (1)$$

Which yields,

$$D = \nabla \cdot \underline{u} = \nabla^2 \phi. \quad (2)$$

We can solve for velocity which yields $\nabla \cdot \underline{u} = D$ by solving for ϕ . Using the Green's function (as derived in class),

$$\phi = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{D(\underline{y})}{|\underline{x} - \underline{y}|} d\underline{y}, \quad (3)$$

from which we get

$$\underline{u} = \nabla \phi = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{D(\underline{y})(\underline{x} - \underline{y})}{|\underline{x} - \underline{y}|^3} d\underline{y}. \quad (4)$$

If you used Green's function directly (derived in class), then you will get 15/20. If you derived the Green's function following the formulae provided as done in class, then you get 20/20.

2. Consider an infinitely long vibrating plate under two fluid layer as shown below. A fluid of density ρ_1 and viscosity μ_1 forms a layer of height d and a fluid of density ρ_2 and viscosity μ_2 forms the layer on top extending to $y \rightarrow \infty$. The plate vibrates with speed $U_0 \sin(\Omega t)$ as shown. Gravity acts downwards. Pressure at the interface of two fluids is p_0 .

- (a) Show that the flow is unidirectional.

Answer:- Since the layers and plate are infinitely long in x , $\frac{\partial}{\partial x}$ for any quantity is zero. From continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \implies \frac{\partial v}{\partial y} = 0. \quad (5)$$

Due to hard-wall condition at $y = 0$, $v = 0$. The flow is unidirectional.

- (b) Simplify the equations governing the flow velocity.

Answer:- Since $v = 0$ and $\frac{\partial}{\partial x} = 0$, the x momentum equation yields,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}. \quad (6)$$

Let bottom layer be labelled as 1, and top layer labeled as 2. Using labels in subscript

$$\frac{\partial u_i}{\partial t} = \nu_i \frac{\partial^2 u_i}{\partial y^2}. \quad (7)$$

Here i is not an index! The y momentum equation simplifies to

$$\frac{\partial p_i}{\partial y} = -\rho_i g. \quad (8)$$

- (c) Derive all the boundary/interface conditions required to solve the flow velocity.

Answer:- At the interface, the kinematic interface condition gives

$$u_1(y = d, t) = u_2(y = d, t). \quad (9)$$

The dynamic interface condition yields,

$$\underline{T} \cdot \hat{n} \Big|_1 = \underline{T} \cdot \hat{n} \Big|_2. \quad (10)$$

This yields,

$$p_1(y = d) = p_2(y = d), \quad \mu_1 \frac{\partial u_1}{\partial y} \Big|_{y=d} = \mu_2 \frac{\partial u_2}{\partial y} \Big|_{y=d}. \quad (11)$$

At $y = 0$, we have the no-slip boundary condition, $u_1(y = 0, t) = U_0 \sin(\Omega t)$. At $y \rightarrow \infty$, we take the flow decays and $u_2(y \rightarrow \infty, t) = 0$.

- (d) Derive the pressure variation in the whole region above the plate.

Answer:- Solving for pressure,

$$p_1(y) - p_0 = -\rho_1 g(y - d) \text{ for } y < d, \quad (12)$$

and

$$p_2(y) - p_0 = -\rho_2 g(y - d) \text{ for } y > d. \quad (13)$$

- (e) Solve for the complex velocity amplitudes of the oscillatory solution of the governing equations using the above derived boundary/interface conditions.

Answer:- Let $u_1 = \hat{u}_1 e^{i\Omega t}$ and similarly for u_2 . We obtain the solutions,

$$\hat{u}_1 = Ae^{\sqrt{\eta_1}y} + Be^{-\sqrt{\eta_1}y} \quad (14)$$

and

$$\hat{u}_2 = Ce^{\sqrt{\eta_2}y} + De^{-\sqrt{\eta_2}y} \quad (15)$$

for $\eta_1 = \frac{i\Omega}{\nu_1}$ and similarly for η_2 . Since $u_2(y \rightarrow \infty) = 0$, we get $C = 0$. The interface kinematic and dynamic conditions yield,

$$Ae^{\sqrt{\eta_1}d} + Be^{-\sqrt{\eta_1}d} = De^{-\sqrt{\eta_2}d} \quad (16)$$

$$\mu_1\sqrt{\eta_1}(Ae^{\sqrt{\eta_1}d} - Be^{-\sqrt{\eta_1}d}) = -\mu_2\sqrt{\eta_2}De^{-\sqrt{\eta_2}d}. \quad (17)$$

At $y = 0$, we obtain $A + B = U_0$. Solving for A, B, D we obtain,

$$A = \frac{U_0 e^{-\sqrt{\eta_1}d} \left(1 - \sqrt{\frac{\rho_2 \mu_2}{\rho_1 \mu_1}}\right)}{e^{\sqrt{\eta_1}d} + e^{-\sqrt{\eta_1}d} + \sqrt{\frac{\rho_2 \mu_2}{\rho_1 \mu_1}}(e^{\sqrt{\eta_1}d} - e^{-\sqrt{\eta_1}d})}, \quad (18)$$

$$B = \frac{U_0 e^{\sqrt{\eta_1}d} \left(1 + \sqrt{\frac{\rho_2 \mu_2}{\rho_1 \mu_1}}\right)}{e^{\sqrt{\eta_1}d} + e^{-\sqrt{\eta_1}d} + \sqrt{\frac{\rho_2 \mu_2}{\rho_1 \mu_1}}(e^{\sqrt{\eta_1}d} - e^{-\sqrt{\eta_1}d})}, \quad (19)$$

$$D = \frac{2U_0}{e^{\sqrt{\eta_1}d} + e^{-\sqrt{\eta_1}d} + \sqrt{\frac{\rho_2 \mu_2}{\rho_1 \mu_1}}(e^{\sqrt{\eta_1}d} - e^{-\sqrt{\eta_1}d})}. \quad (20)$$

The above expressions yield,

$$\frac{\hat{u}_1}{U_0} = \frac{\cosh(\sqrt{\eta_1}(y-d)) - \sqrt{\frac{\rho_2 \mu_2}{\rho_1 \mu_1}} \sinh(\sqrt{\eta_1}(y-d))}{\cosh(\sqrt{\eta_1}d) + \sqrt{\frac{\rho_2 \mu_2}{\rho_1 \mu_1}} \sinh(\sqrt{\eta_1}d)} \quad (21)$$

and

$$\frac{\hat{u}_2}{U_0} = \frac{e^{-\sqrt{\eta_2}(y-d)}}{\cosh(\sqrt{\eta_1}d) + \sqrt{\frac{\rho_2 \mu_2}{\rho_1 \mu_1}} \sinh(\sqrt{\eta_1}d)} \quad (22)$$

- (f) Simplify the above solution for $\rho_2 = \frac{\rho_1}{K}$ and $\mu_2 = K\mu_1$. State the full real oscillatory solution.

Answer:- For $\rho_2 = \frac{\rho_1}{K}$ and $\mu_2 = K\mu_1$, we get $\sqrt{\frac{\rho_2 \mu_2}{\rho_1 \mu_1}} = 1$. The complex amplitudes simplify to

$$\frac{\hat{u}_1}{U_0} = \frac{e^{-\sqrt{\eta_1}(y-d)}}{e^{\sqrt{\eta_1}d}} = e^{-\sqrt{\eta_1}y}, \quad (23)$$

and

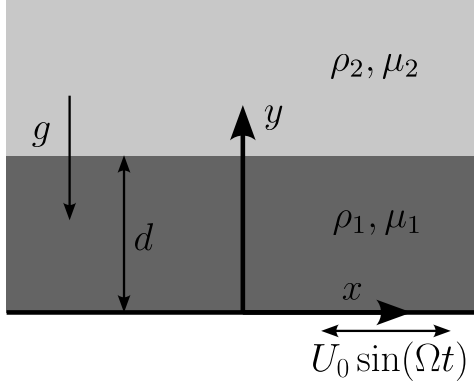
$$\frac{\hat{u}_2}{U_0} = e^{\sqrt{\eta_1}d(\frac{1}{K}-1)-\sqrt{\eta_2}y}. \quad (24)$$

Since $U_0 \sin \Omega t = \text{Im}(U_0 e^{i\Omega t})$, we obtain,

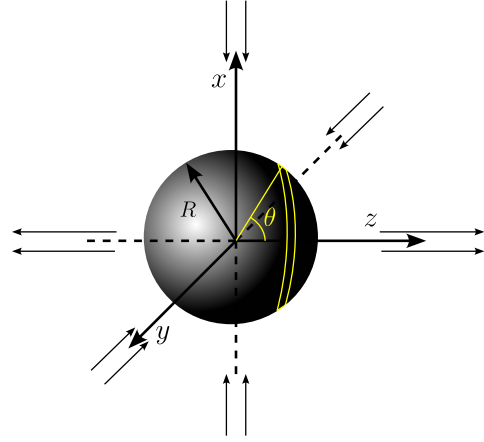
$$u_1 = U_0 e^{-\sqrt{\frac{\Omega}{2\nu_1}}y} \sin\left(\Omega t - \sqrt{\frac{\Omega}{2\nu_1}}y\right), \quad (25)$$

and

$$u_2 = U_0 e^{-\sqrt{\frac{\Omega}{2\nu_2}} y} e^{d\sqrt{\frac{\Omega}{2\nu_1}} \left(\frac{1+K}{K}\right)} \sin \left(\Omega t - \sqrt{\frac{\Omega}{2\nu_2}} y + d\sqrt{\frac{\Omega}{2\nu_1}} \left(\frac{1+K}{K}\right) \right). \quad (26)$$



Problem 1



Problem 2

3. An extensional flow symmetric about z axis is defined by flow converging on z axis in xy plane. The velocity field is given by,

$$\underline{u} = -E \left(x\hat{i} + y\hat{j} - 2z\hat{k} \right),$$

where E is some parameter. The flow tends to extend the z axis, hence the name extensional flow.

- (a) Calculate the velocity gradient for this flow and its symmetric and anti-symmetric components.

Answer:- The velocity gradient tensor is,

$$\nabla \underline{u} = -E \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad (27)$$

Clearly, $\underline{\underline{\Omega}} = 0$. The flow is irrotational.

- (b) Consider a sphere of radius R in extensional flow of a fluid with viscosity μ and density ρ . State the velocity at $r \rightarrow \infty$ in spherical coordinate system.

Answer:- At $r \rightarrow \infty$, velocity approaches free stream. Using the transformations

$$\hat{i} = \hat{e}_r \sin \theta \cos \phi + \hat{e}_\theta \cos \theta \cos \phi - \hat{e}_\phi \sin \phi, \quad (28)$$

$$\hat{j} = \hat{e}_r \sin \theta \sin \phi + \hat{e}_\theta \cos \theta \sin \phi + \hat{e}_\phi \cos \phi, \quad (29)$$

$$\hat{k} = \hat{e}_r \cos \theta - \hat{e}_\theta \sin \theta, \quad (30)$$

and

$$x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta \quad (31)$$

we obtain for $\underline{u}_\infty$

$$\underline{u}_\infty = -Er (1 - 3 \cos^2 \theta) \hat{e}_r - 3Er \sin \theta \cos \theta \hat{e}_\theta. \quad (32)$$

(c) What is the form of stream-function which satisfies the above velocity at $r \rightarrow \infty$?

Answer:- Using

$$u_r = -Er (1 - 3 \cos^2 \theta) = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}, \quad (33)$$

we obtain

$$\psi = Er^3 \eta (1 - \eta^2) \quad (34)$$

for $r \rightarrow \infty$, where $\eta = \cos \theta$. Hence,

$$\psi = g(r) \eta (1 - \eta^2) \quad (35)$$

where $\lim_{r \rightarrow \infty} g(r) = Er^3$.

(d) Using the above form, solve for the stream-function for the extensional flow past the sphere.

Answer:- Converting the governing equation for ψ in (r, η) for simplicity (this is not a mandatory step, but makes algebra much more convenient)

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{1 - \eta^2}{r^2} \frac{\partial^2}{\partial \eta^2} \quad (36)$$

and substituting the above form of ψ , we obtain,

$$g'''' - \frac{12g''}{r^2} + \frac{24g'}{r^3} = 0 \quad (37)$$

Recognizing this as Euler-Cauchy equation, we use $g = r^n$ and obtain,

$$n(n^3 - 6n^2 - n + 30) = 0. \quad (38)$$

Using given data, the cubic can be easily factorized to obtain,

$$n(n - 3)(n - 5)(n + 2) = 0. \quad (39)$$

Hence the general solution is,

$$g(r) = A + Br^3 + Cr^5 + \frac{D}{r^2}. \quad (40)$$

At $r \rightarrow \infty$, $g(r)$ behaves as r^3 hence $C = 0$ and $B = E$. Boundary conditions at the sphere $r = R$ are

$$u_r = 0 \implies g(R) = 0, u_\theta = 0 \implies g'(R) = 0. \quad (41)$$

Using these, we obtain $D = \frac{3ER^5}{2}$ and $A = -\frac{5ER^3}{2}$ and ψ as,

$$\psi = -ER^3 \left(\frac{5}{2} - \left(\frac{r}{R} \right)^3 - \frac{3}{2} \left(\frac{R}{r} \right)^5 \right) \eta (1 - \eta^2). \quad (42)$$

(e) Using the above solution, solve for the pressure force on the sphere.

Answer:- The velocity components are,

$$u_r = E \left(r - \frac{5R^3}{2r^2} + \frac{3R^6}{2r^5} \right) (3 \cos^2 \theta - 1), \quad u_\theta = -3E \left(r - \frac{R^5}{r^4} \right) \sin \theta \cos \theta. \quad (43)$$

Using $\frac{\partial p}{\partial r} = \mu (\nabla^2 \underline{u}) \cdot \hat{e}_r$, we obtain,

$$p(R) = p_\infty - f(R)(3 \cos^2 \theta - 1), \quad (44)$$

where it is not necessary to find $f(r)$ since the flow is directed towards the sphere such that there is no force on the sphere (symmetric in all axis). From above expression, we note that

$$\int_0^\pi (3 \cos^2 \theta - 1) \sin \theta d\theta = 0. \quad (45)$$

This shows that the total pressure force is 0. Note that there is problem in $\nabla^2 \underline{u}$ expression in $\hat{e}_\theta$ direction. If your calculation was wrong/hindered due to that expression, you get full points.

(f) What is the total force on the sphere due to the flow?

Answer:- Given how the flow is directed towards the sphere, the sphere can remain at rest. Hence the total force is 0.