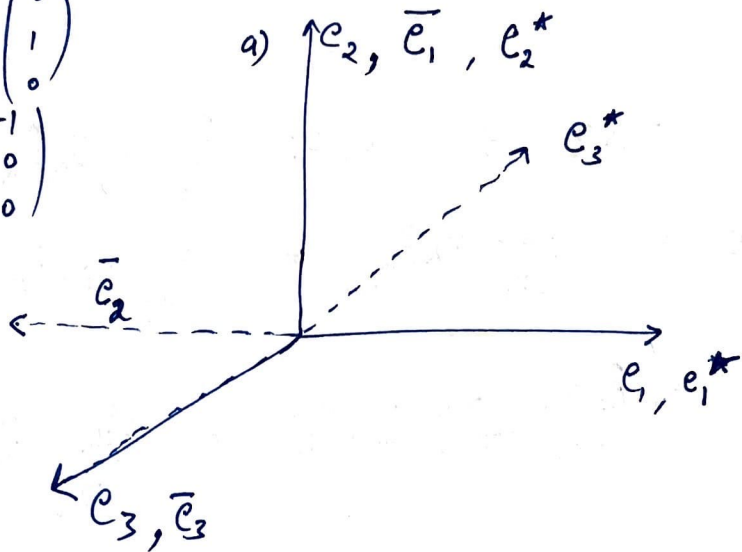


PRACTICE SHEET - 1

SOLUTIONS

$$\textcircled{1} \text{ b) } \begin{aligned} \bar{e}_1 &= e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ \bar{e}_2 &= -e_1 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \\ \bar{e}_3 &= e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$



$$e_1^* = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2^* = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3^* = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\text{c) } [\bar{e}_1, \bar{e}_2, \bar{e}_3] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [\bar{a}_{ij}] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \bar{a}_{ij} = \bar{e}_i \cdot \bar{e}_j = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[e_1^*, e_2^*, e_3^*] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = [a_{ij}^*] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$a_{ij}^* = e_i \cdot e_j^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\text{d) } \bar{a}_{ik} \bar{a}_{jk} = [\bar{a}_{ij}] [\bar{a}_{ij}]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{SIMILARLY} \quad [a_{ij}^*] [a_{ij}^*]^T = \underline{\underline{I}}$$

$$e) \quad \|\bar{a}_{ij}\| = -(-1)(1) = 1, \quad \|a_{ij}^*\| = 1(1)(-1) = -1$$

(ROTATION MATRIX) (REFLECTION MATRIX)

$$\begin{aligned} \textcircled{2} \quad a) \quad \underline{\omega} &= \nabla \times \underline{u} \Rightarrow \omega_i \hat{e}_i = \hat{e}_j \frac{\partial}{\partial x_j} \times u_k \hat{e}_k = \frac{\partial u_k}{\partial x_j} \hat{e}_j \times \hat{e}_k \\ \nabla \cdot \underline{\omega} &= \hat{e}_i \frac{\partial}{\partial x_i} (\omega_k \hat{e}_k) = \frac{\partial \omega_k}{\partial x_i} \hat{e}_i \cdot \hat{e}_k = \frac{\partial \omega_k}{\partial x_i} \delta_{ik} = \frac{\partial \omega_i}{\partial x_i} \\ &= \hat{e}_i \frac{\partial}{\partial x_i} \left(\frac{\partial u_k}{\partial x_j} \hat{e}_j \times \hat{e}_k \right) = \frac{\partial^2 u_k}{\partial x_i \partial x_j} \hat{e}_i \cdot (\hat{e}_j \times \hat{e}_k) = \epsilon_{ijk} \frac{\partial^2 u_k}{\partial x_i \partial x_j} \\ &= \epsilon_{jik} \frac{\partial^2 u_k}{\partial x_i \partial x_j} = -\epsilon_{ijk} \frac{\partial^2 u_k}{\partial x_i \partial x_j} \Rightarrow \nabla \cdot \underline{\omega} = 0 \end{aligned}$$

$$\begin{aligned} b) \quad \nabla \times (\nabla \phi) &= \hat{e}_i \frac{\partial}{\partial x_i} \times \hat{e}_j \left(\frac{\partial}{\partial x_j} \phi \right) = \frac{\partial^2 \phi}{\partial x_i \partial x_j} \hat{e}_i \times \hat{e}_j \\ &= -\frac{\partial^2 \phi}{\partial x_i \partial x_j} \hat{e}_j \times \hat{e}_i \quad (\text{SWITCH } i \text{ and } j) \\ &= -\frac{\partial^2 \phi}{\partial x_i \partial x_j} \hat{e}_i \times \hat{e}_j \quad (\hat{e}_i \times \hat{e}_j = -\hat{e}_j \times \hat{e}_i) \\ &\Rightarrow \nabla \times (\nabla \phi) = 0 \end{aligned}$$

$$\begin{aligned} c) \quad \nabla \times (\nabla \times \underline{u}) &= \hat{e}_i \frac{\partial}{\partial x_i} \times \left(\hat{e}_j \frac{\partial}{\partial x_j} \times \hat{e}_k u_k \right) \\ &= \hat{e}_i \times \hat{e}_j \times \hat{e}_k \frac{\partial^2 u_k}{\partial x_i \partial x_j} = \epsilon_{ljk} \hat{e}_i \times \hat{e}_l \frac{\partial^2 u_k}{\partial x_i \partial x_j} \\ &= \hat{e}_m \epsilon_{ljk} \epsilon_{mil} \frac{\partial^2 u_k}{\partial x_i \partial x_j} = \hat{e}_m \left(\epsilon_{ljk} \epsilon_{lmi} \frac{\partial^2 u_k}{\partial x_i \partial x_j} \right) \\ &= \hat{e}_m \left(\delta_{jm} \delta_{ki} - \delta_{ji} \delta_{km} \right) \frac{\partial^2 u_k}{\partial x_i \partial x_j} = \frac{\partial^2 u_i}{\partial x_i \partial x_m} \hat{e}_m - \frac{\partial^2 u_m}{\partial x_i \partial x_i} \hat{e}_m = \nabla (\nabla \cdot \underline{u}) - \nabla^2 \underline{u} \end{aligned}$$

$$\begin{aligned}
 d) \quad \underline{u} \times \underline{\omega} &= \hat{e}_i u_i \times \hat{e}_j \omega_j = \hat{e}_i u_i \times \left(\hat{e}_j \frac{\partial}{\partial x_j} \times \hat{e}_k u_k \right) \\
 &= \hat{e}_i \times \hat{e}_j \times \hat{e}_k u_i \frac{\partial u_k}{\partial x_j} = \hat{e}_i \times \hat{e}_l \varepsilon_{ljk} u_i \frac{\partial u_k}{\partial x_j} \\
 &= \hat{e}_m \varepsilon_{mil} \varepsilon_{ljk} u_i \frac{\partial u_k}{\partial x_j} = \hat{e}_m \left(\delta_{im} \delta_{lk} - \delta_{jk} \delta_{ml} \right) u_i \frac{\partial u_k}{\partial x_j} \\
 &= u_k \frac{\partial u_k}{\partial x_j} \hat{e}_j - u_i \frac{\partial u_m}{\partial x_j} \hat{e}_m \\
 &= \nabla \left(\frac{|\underline{u}|^2}{2} \right) - \underline{u} \cdot \nabla (\underline{u})
 \end{aligned}$$

$$\textcircled{3} \quad a) \quad \nabla |\underline{r}| = \hat{e}_i \frac{\partial}{\partial x_i} \sqrt{x_k x_k} \quad r^2 = x_k x_k$$

$$\nabla(r^2) = \hat{e}_i \frac{\partial}{\partial x_i} \cancel{r^2} = \hat{e}_i \frac{\partial}{\partial x_i} (x_k x_k)$$

$$\Rightarrow 2r \nabla r = 2x_k \hat{e}_i \delta_{ki} \Rightarrow \nabla r = \frac{x_i \hat{e}_i}{r} = \hat{\underline{r}} \quad (\text{UNIT VECTOR})$$

$$b) \quad \nabla \left(\frac{1}{r} \right) = - \frac{\nabla r}{r^2} = - \frac{\hat{\underline{r}}}{r^2}$$

$$c) \quad \nabla \cdot \left(\nabla \left(\frac{1}{r} \right) \right) = - \nabla \cdot \left(\frac{\hat{\underline{r}}}{r^2} \right) = - \frac{\nabla \cdot \hat{\underline{r}}}{r^2} + \frac{2}{r^3} (\nabla r) \cdot \hat{\underline{r}}$$

$$- \nabla \cdot \left(\frac{\hat{\underline{r}}}{r^2} \right) = - \nabla \cdot \left(\frac{\underline{r}}{r^3} \right) = - \frac{1}{r^3} \nabla \cdot \underline{r} + \frac{3}{r^4} (\nabla r) \cdot \underline{r}$$

$$\begin{aligned}
 \nabla \cdot \underline{r} &= \varepsilon_{ij} \frac{\partial x_j}{\partial x_i} = 3 \\
 &= 3 - \frac{3}{r^3} + \frac{3}{r^4} \frac{\underline{r}}{r} \cdot \underline{r} = 0
 \end{aligned}$$

$$\begin{aligned}
 d) \quad \nabla \times \underline{r} &= \hat{e}_i \varepsilon_{ijk} \frac{\partial}{\partial x_j} x_k = \hat{e}_i \varepsilon_{ijk} \frac{\partial x_j}{\partial x_k} = - \hat{e}_i \varepsilon_{ikj} \frac{\partial x_k}{\partial x_j} = - \hat{e}_i \varepsilon_{ijk} \frac{\partial x_j}{\partial x_k} \\
 &= 0
 \end{aligned}$$

$$(4) \quad S_{ij} = + S_{ji} \quad , \quad T_{ij} = \underbrace{\frac{1}{2} (T_{ij} + T_{ji})}_{T_{ij}^s} + \underbrace{\frac{1}{2} (T_{ij} - T_{ji})}_{T_{ij}^a}$$

$$a) \quad S_{ij} T_{ij}^a = - S_{ji} T_{ji}^a = - S_{ij} T_{ij}^a = 0$$

$(T_{ij}^a = -T_{ji}^a)$
(SWITCH DUMMY INDEXES)

$$b) \quad S_{ij} T_{ij} = S_{ij} T_{ij}^s + S_{ij} T_{ij}^a = S_{ij} T_{ij}^s$$

$$(5) \quad \nabla_s \phi = \nabla \phi - (\hat{n} \cdot \nabla \phi) \hat{n} \quad \nabla_s \phi \cdot \hat{n} = (\nabla \phi) \cdot \hat{n} - (\hat{n} \cdot \nabla \phi) (\underbrace{\hat{n} \cdot \hat{n}}_1)$$

USING INDEX NOTATION

$$\hat{n} \cdot (\nabla_s \phi) = \left(\hat{e}_i \frac{\partial \phi}{\partial x_i} - (\hat{n}_j \frac{\partial \phi}{\partial x_j}) \hat{n}_i \hat{e}_i \right) \cdot \hat{n}_k \hat{e}_k$$

$$= \frac{\partial \phi}{\partial x_i} \hat{n}_k \delta_{ik} - \hat{n}_j \frac{\partial \phi}{\partial x_j} \hat{n}_i \hat{e}_i \cdot \hat{e}_k \hat{n}_k = \frac{\partial \phi}{\partial x_i} \hat{n}_i - \hat{n}_j \frac{\partial \phi}{\partial x_j} (\underbrace{\hat{n}_k \hat{n}_k}_1)$$

= 0

$$(6) \quad V(r) = \frac{r^4}{4} \quad \underline{F}_x = - \frac{\partial V}{\partial x} = - \hat{e}_i \frac{\partial V}{\partial x_i} = - \hat{e}_i \frac{\partial}{\partial x_i} \left(\frac{r^4}{4} \right)$$

$$r^2 = (x_j - y_j)(x_j - y_j) \quad = - \hat{e}_i r^3 \frac{\partial r}{\partial x_i}$$

$$2r \frac{\partial r}{\partial x_i} = 2(x_i - y_i) \frac{\partial x_i}{\partial x_i} \Rightarrow \frac{\partial r}{\partial x_i} = \frac{x_i - y_i}{r} \Rightarrow \underline{F}_x = - \hat{e}_i r^3 \left(\frac{x_i - y_i}{r} \right)$$

SIMILARLY

$$\underline{F}_y = - \hat{e}_i r^3 \frac{\partial r}{\partial y_i} \quad 2r \frac{\partial r}{\partial y_i} = -2(x_i - y_i) \frac{\partial y_i}{\partial y_i} = - \frac{(x_i - y_i)}{r}$$

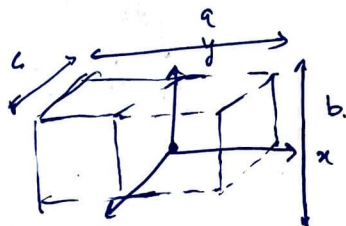
$$\Rightarrow \underline{F}_y = - r^3 \left(\frac{y - x}{r} \right) = - \underline{F}_x$$

⑦ GAUSS' DIVERGENCE $\oint_S \underline{A} \cdot \hat{n} \, dS = \int_V (\nabla \cdot \underline{A}) \, dV = \int_V 3 \, dV$

$$\nabla \cdot \underline{A} = \epsilon_i \frac{\partial}{\partial x_i} A_i \hat{e}_i = \frac{\partial x_i}{\partial x_i} = 3$$

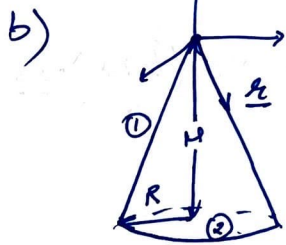
$$\Rightarrow V = \frac{1}{3} \oint_S \underline{A} \cdot \hat{n} \, dS$$

a) FOR A RECTANGULAR PARALLELOPIPED



$$V = \frac{1}{3} \left(2 \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{c}{2}}^{\frac{c}{2}} \left(\frac{a}{2} \hat{i} + y \hat{j} + z \hat{k} \right) \cdot \hat{i} \, dy \, dz + 2 \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \left(x \hat{i} + y \hat{j} + \frac{c}{2} \hat{k} \right) \cdot \hat{k} \, dx \, dy + 2 \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \left(x \hat{i} + \frac{b}{2} \hat{j} + z \hat{k} \right) \cdot \hat{j} \, dx \, dz \right)$$

$$= \frac{1}{3} (3abc) = abc$$

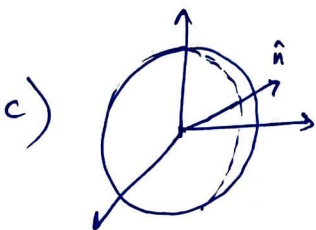


$$V = \frac{1}{3} \left(\int_{(1)} \underline{A} \cdot \hat{n} \, dS + \int_{(2)} \underline{A} \cdot \hat{n} \, dS \right)$$

$$(\underline{A} \cdot \hat{n})_1 = 0, \quad (\underline{A} \cdot \hat{n})_2 = H, \quad dS = |\underline{A}| \sin \theta \, d\phi$$

$$V = \frac{1}{3} \left(0 + \int_0^{2\pi} \int_0^R \tilde{A} \, d\tilde{A} \, d\phi \right)$$

$$= \frac{\pi R^2 H}{3}$$



$$\underline{A} \cdot \hat{n} = R \quad dS = R^2 \sin \theta \, d\theta \, d\phi$$

$$\Rightarrow V = \frac{1}{3} \int_0^{2\pi} \int_0^\pi R \sin \theta \, d\theta \, d\phi = \frac{(2\pi)(2)R^3}{3} = \frac{4\pi R^3}{3}$$

$$\begin{aligned}
 \textcircled{8} \quad \iint_S \underline{F} \cdot \hat{n} ds &= \int_{\text{CUBE}} (\nabla \cdot \underline{F}) dV & \nabla \cdot \underline{F} &= e^x + e^y + e^z \\
 &= \int_{-1}^1 dx \int_{-1}^1 dy \int_{-1}^1 dz (e^x + e^y + e^z) & &= 4 \int_{-1}^1 e^x dx + 4 \int_{-1}^1 e^y dy + 4 \int_{-1}^1 e^z dz \\
 & & &= 12 \left(e - \frac{1}{e} \right)
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{9} \quad & \text{Diagram: A shaded volume V in a 3D coordinate system with axes x, y, z. The boundary is C. A point A is marked inside. A vector $\underline{r}$ points from A to a point on the boundary C. A normal vector $\hat{n}$ points outwards from the surface.} \\
 & \frac{1}{2} \oint_C (\underline{r} \times \hat{t}) \cdot \hat{k} d\ell \\
 & (\underline{r} \times \hat{t}) \cdot \hat{k} = \hat{e}_1 \varepsilon_{123} r_2 t_3 \cdot k_1 \hat{e}_1 = \varepsilon_{123} k_1 r_2 t_3 \\
 & = \hat{t} \cdot (\hat{k} \times \underline{r}) \\
 & \frac{1}{2} \oint_C \hat{t} \cdot (\hat{k} \times \underline{r}) d\ell = \frac{1}{2} \oint_C (\hat{k} \times \underline{r}) \cdot d\underline{\ell} = \frac{1}{2} \int_S \nabla \times (\hat{k} \times \underline{r}) \cdot \underline{ds} \\
 & = \frac{1}{2} \int_S \nabla \times (\hat{k} \times \underline{r}) \cdot \hat{n} ds \quad (\hat{n} = \hat{k}) \\
 & \nabla \times (\underline{r} \times \underline{r}) = (\delta_{12} \delta_{3m} - \delta_{13} \delta_{2m}) k_1 \delta_{mj} = k_1 \delta_{3j} - k_2 \delta_{1j} \\
 & = 3k_1 - k_1 = 2k_1 \\
 & \boxed{k_1 = \delta_{13}} \quad k_i \hat{n}_i = 1
 \end{aligned}$$

$$\Rightarrow \frac{1}{2} \oint_C (\underline{r} \times \hat{t}) \cdot \hat{k} d\ell = \frac{1}{2} \int_S 2(\hat{k} \cdot \hat{k}) ds = S$$