

PRACTICE-2

SOLUTIONS

$$\textcircled{1} a) \quad x_j \hat{e}_j \times \left(\frac{\partial (\rho u_i)}{\partial t} \hat{e}_i + \frac{\partial (\rho u_k u_i)}{\partial x_k} \hat{e}_i \right) = f_i \hat{e}_i + \frac{\partial (\tau_{ik})}{\partial x_k} \hat{e}_i$$

→
ASSUMING CARTESIAN

$$\Rightarrow \quad x_j \frac{\partial (\rho u_i)}{\partial t} \hat{e}_j \times \hat{e}_i + x_j \frac{\partial (\rho u_k u_i)}{\partial x_k} \hat{e}_j \times \hat{e}_i = x_j f_i \hat{e}_j \times \hat{e}_i + x_j \frac{\partial \tau_{ik}}{\partial x_k} \hat{e}_j \times \hat{e}_i$$

↑
EULERIAN COORDINATE

$$\Rightarrow \quad \frac{\partial (\rho u_i x_j)}{\partial t} \epsilon_{ijl} \hat{e}_l + x_j \frac{\partial (\rho u_k u_i)}{\partial x_k} \epsilon_{ijl} \hat{e}_l = x_j f_i \epsilon_{ijl} \hat{e}_l + x_j \frac{\partial \tau_{ik}}{\partial x_k} \epsilon_{ijl} \hat{e}_l$$

b) REPLACING ϕ by $\epsilon_{ijk} x_j u_k$.

$$\int \frac{\partial (\epsilon_{ijk} x_j u_k \rho)}{\partial t} dv + \int \frac{\partial (\rho u_k \epsilon_{ijk} x_j u_k)}{\partial x_k} dv = \int \epsilon_{ijk} x_j f_k dv + \frac{\partial}{\partial x_k} \oint (\underline{x} \times \underline{\tau}) \cdot \hat{n} ds$$

$$\oint (\underline{x} \times \underline{\tau}) \cdot \hat{n} ds = \int_V \nabla \cdot (\underline{x} \times \underline{\tau}) dv$$

$$= \int_V \frac{\partial}{\partial x_k} (\epsilon_{ijk} x_j \tau_{kl}) dv$$

$$\Rightarrow \frac{\partial}{\partial t} \left(\varepsilon_{ijk} x_j (p_{uk}) \right) + \frac{\partial}{\partial x_l} \left(\varepsilon_{ijk} x_j u_k (p_{ul}) \right) = \varepsilon_{ijk} x_j f_k$$

$$+ \frac{\partial}{\partial x_l} \left(\varepsilon_{ijk} x_j T_{kl} \right)$$

for $x_i = 0$

— (2)

c) SUBTRACTING (1) FROM (2)

$$- \hat{e}_i \varepsilon_{lji} \left(\frac{\partial}{\partial x_k} (p_{ul} u_k) \right) x_j + \frac{\partial}{\partial x_l} \left(\varepsilon_{ijk} x_j u_k (p_{ul}) \right) \hat{e}_i$$

$$= \hat{e}_i \varepsilon_{ijk} \frac{\partial}{\partial x_l} (x_j T_{kl}) - \varepsilon_{lji} x_j \frac{\partial T_{ik}}{\partial x_k} \hat{e}_l$$

SWITCHING DUMMY INDICES

$$\Rightarrow - \varepsilon_{ijk} \frac{\partial}{\partial x_l} (p_{uk} u_l) x_j + \varepsilon_{ijk} \frac{\partial}{\partial x_l} (p_{ul} u_k x_j)$$

$$= \varepsilon_{ijk} \frac{\partial}{\partial x_l} (x_j T_{kl}) - \varepsilon_{ijk} x_j \frac{\partial T_{kl}}{\partial x_l}$$

$$p_{ul} u_k \varepsilon_{ijk} \frac{\partial x_j}{\partial x_l} = \varepsilon_{ijk} T_{kl} \frac{\partial x_j}{\partial x_l}$$

$$\varepsilon_{ijk} p_{ul} u_k = \varepsilon_{ijk} T_{kl}$$

$$\varepsilon_{ikl} p_{ul} u_k = \varepsilon_{ikl} u_k u_l = - \varepsilon_{ikl} u_k u_l = 0$$

$$\Rightarrow \varepsilon_{ikl} T_{kl} = 0 \Rightarrow \boxed{T_{kl} = T_{lk}}$$

② a) $\underline{u}(\underline{x}) = \underline{u}$ $\underline{u}(\underline{x} + \underline{h}) = \underline{u}(\underline{x}) + (\nabla \underline{u})|_{\underline{x}} \cdot \underline{h} + \frac{1}{2} \nabla (\nabla \underline{u}) : \underline{h} \otimes \underline{h} + \dots$

UP TO FIRST ORDER $\underline{u}(\underline{x} + \underline{h}) = \underline{u} + (\nabla \underline{u})|_{\underline{x}} \cdot \underline{h}$

$$u_i(\underline{x} + \underline{h}) = u_i + h_j \frac{\partial u_i}{\partial x_j} \Rightarrow \delta u_i = h_j \frac{\partial u_i}{\partial x_j}$$

⑥

$$= \frac{h_j}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{h_j}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)$$

③

$$\frac{1}{2} \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$= \lambda_{ij} \delta_{ij}$$

RATE OF SEPARATION OF MATERIAL POINTS

$$\delta \underline{u} = \delta u_i \hat{e}_i$$

$$\delta u_i = \frac{h_j}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = h_j \lambda_{ij}$$

$$h_i = (h_1, 0, 0)$$

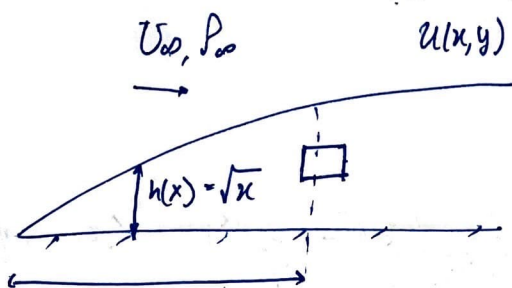
$$\Rightarrow \delta u_i = h_i \lambda_{ii} \Rightarrow \delta \underline{u} = \begin{pmatrix} h_1 \lambda_1 \\ 0 \\ 0 \end{pmatrix} \cdot \frac{\underline{x} - \underline{x}_0}{\ell} \cdot \left(\frac{d\ell}{dt} - \lambda_1 \ell \right)$$

④

$$\delta u_i = \frac{h_j}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) = h_j \Omega_{ij} = -\frac{1}{2} \epsilon_{ijk} h_j \omega_k$$

$$\Rightarrow \delta \underline{u} = \frac{1}{2} \underline{\omega} \times \underline{h}$$

⑤



$$u(x, y) = U_\infty (2\eta - \eta^2), \quad \eta = \frac{y}{\sqrt{x}}, \quad \frac{\partial \eta}{\partial x} = -\frac{y}{2x^{3/2}}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad = -\frac{1}{2x}$$

$$\frac{\partial u}{\partial x} = 2U_\infty \frac{\partial \eta}{\partial x} - 2U_\infty \eta \frac{\partial \eta}{\partial x}$$

$$= 2U_\infty (1 - \eta) \frac{\partial \eta}{\partial x}$$

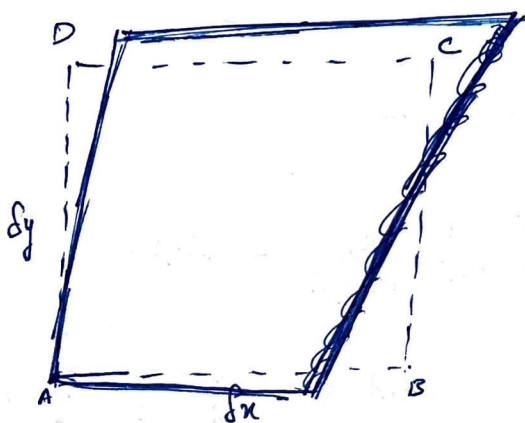
$$= 2U_\infty (1 - \eta) \left(-\frac{1}{2x} \right)$$

$$\int_0^y dv = \int_0^y -\frac{\partial v}{\partial x} dy = \int_0^y 2U_\infty (1-\eta) \frac{\eta}{2x} dy$$

$$= \frac{U_\infty}{x} \int_0^y \left(\frac{y}{\sqrt{x}} - \frac{y^2}{x} \right) dy$$

$$v = \frac{U_\infty y^2}{2x^{3/2}} - \frac{U_\infty y^3}{3x^2}$$

9)
$$v = \frac{U_\infty}{\sqrt{x}} \left(\frac{\eta^2}{2} - \frac{\eta^3}{3} \right)$$



$$u_A = u(L, \sqrt{L})$$

$$u_A = u(x, y), \quad u_B = u + \left(\frac{\partial u}{\partial x} \right)_A \delta x$$

$$u_C = u + \left. \frac{\partial u}{\partial x} \right|_A \delta x + \left. \frac{\partial u}{\partial y} \right|_A \delta y$$

$$u_D = u + \left. \frac{\partial u}{\partial y} \right|_A \delta y$$

$$\frac{\partial u}{\partial x} = -\frac{2U_\infty \eta}{x} (1-\eta) < 0$$

$$\frac{\partial u}{\partial y} = 2U_\infty \frac{\partial \eta}{\partial y} - 2U_\infty \eta \frac{\partial \eta}{\partial y} = 2U_\infty \left(\frac{1}{\sqrt{x}} - \frac{y}{\sqrt{x}} \frac{1}{\sqrt{x}} \right) > 0$$

$$= 2U_\infty \left(\frac{1}{\sqrt{x}} - \frac{y}{x} \right)$$

$$\frac{\partial v}{\partial x} = -\frac{3}{2} U_\infty \frac{y^2}{x^{5/2}} + \frac{2U_\infty y^3}{3x^2}$$

$$= \left(\frac{2y^3}{3x^2} - \frac{3y^2}{4x^{5/2}} \right) U_\infty = \frac{U_\infty}{(\sqrt{x})^{3/2}} \left(\frac{2}{3} \eta^3 - \frac{3}{4} \eta^2 \right) < 0 \quad (\eta < 1)$$

$$\frac{\partial v}{\partial y} = \frac{U_\infty y}{x^{3/2}} - \frac{U_\infty y^2}{x^2} = \frac{U_\infty}{x} (1 - \eta^2) > 0$$

$$b) \nabla u = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

c) CALCULATE

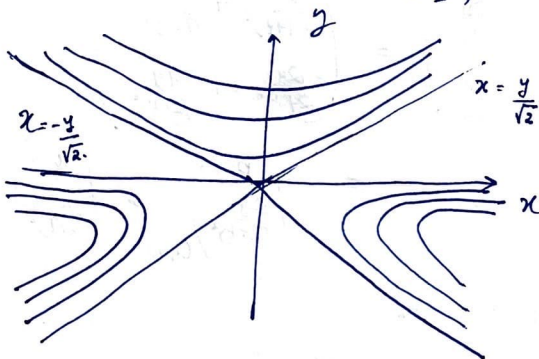
$$\frac{1}{2}(\nabla u + \nabla u^T) \quad \& \quad \frac{1}{2}(\nabla u - \nabla u^T)$$

$$④ \quad u = 3(x^2 - y^2), \quad v = -6xy$$

$$\frac{dy}{dx} = \frac{-6xy}{3(x^2 - y^2)} \Rightarrow x^2 dy - y^2 dy = -2xy dx$$

$$x^2 dy + 2xy dx = y^2 dy$$

$$d(x^2 y) = d\left(\frac{y^3}{3}\right) \Rightarrow \boxed{x^2 y = \frac{y^3}{3} + C}$$



$$⑤ \quad u = \frac{x}{1+t}, \quad v = \frac{2y}{2+t}$$

b) STREAKLINE

$$x(t=1) = 1, \quad y(t=1) = 1$$

$$\int_1^x \frac{dx}{x} = \int_1^t \frac{dt}{1+t}$$

$$\boxed{x = \frac{1+t}{1+t}, \quad y = \left(\frac{t+2}{t+2}\right)^2}$$

PARAMETERIZED

EQN (t IS PARAMETER)

a) PATHLINE

$$\int_1^x \frac{dx}{x} = \int_0^t \frac{dt}{1+t}, \quad \int_1^y \frac{dy}{y} = \int_0^t \frac{2dt}{2+t}$$

$$x = 1+t, \quad y = \left(\frac{2+t}{2}\right)^2$$

$$\boxed{y = \left(\frac{x+1}{2}\right)^2}$$

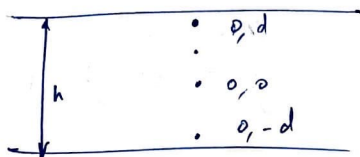
c) STREAMLINE

$$\frac{dy}{dx} = \frac{2y}{x} \frac{1+t}{2+t}$$

$$\Rightarrow x^{t+1} = y^{\frac{t+2}{2}} + C$$

$$(x, y) = (1, 1) \text{ AT } t=0 \Rightarrow \boxed{x=y}$$

⑥



$$u = A \sin(\omega t - kx) \left[1 - \frac{\cosh(ky)}{\cosh(kh)} \right]$$

(UNIDIRECTIONAL) $f(y)$

a) Acc. $= \frac{\partial u}{\partial t} + \underline{u} \cdot \nabla \underline{u} \Rightarrow A \omega \cos(\omega t - kx) f + u \frac{\partial u}{\partial x}$

$$= A \omega \cos(\omega t - kx) f + A \sin(\omega t - kx) f \left[-A k \cos(\omega t - kx) f \right]$$

$$= A f \omega \cos(\omega t - kx) - A^2 k f \sin(\omega t - kx) \cos(\omega t - kx)$$

SUBSTITUTE x, ω, A, y

b) $\frac{dT}{dt} = \frac{\partial T}{\partial t} + \underline{u} \cdot \nabla T = u \frac{\partial T}{\partial x} = 50u$

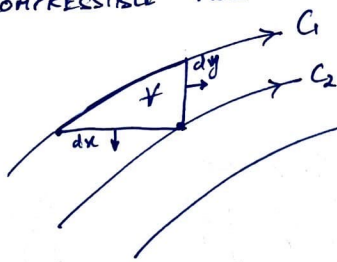
⑦

$$\underline{a} \Big|_{(1,1)} = \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = \begin{pmatrix} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{-x}{(1+t)^2} + \frac{x}{(1+t)^2} \\ \frac{-2y}{(2+t)^2} + \frac{4y}{(2+t)^2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \frac{2y}{(2+t)^2} \end{pmatrix} \Big|_{(1,1)} = \begin{pmatrix} 0 \\ \frac{2}{(2+1)^2} \end{pmatrix}$$

⑧

FOR INCOMPRESSIBLE FLOW



STREAMLINES

CONSIDER CV

APPLYING MASS EQ.

$$\oint \underline{u} \cdot \hat{n} ds = 0$$

$$\Rightarrow \int u dy - \int v dx = 0 \quad \text{LET CURVES } C_i \text{ BE SUCH THAT } f(x,y) = \text{const.}$$

$$\Rightarrow \text{ON } C_1 \quad df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

ALSO IN LIMIT $C_2 \rightarrow C_1 \quad u dy - v dx = 0.$

$$\Rightarrow \frac{\partial f}{\partial x} = -v, \quad \frac{\partial f}{\partial y} = u$$

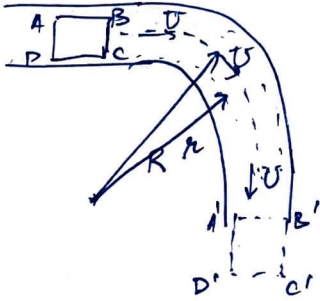
$$\Rightarrow \boxed{f \equiv \psi}$$

$\Rightarrow$ STREAMLINES ARE ISOLINES OF ψ .

⑨ SEE HANDOUTS

⑩

a)



UNIFORM SPEED .

~~FLUID ELEMENT TRANSLATES WITHOUT ROTATION~~
 ~~$\nabla \times \mathbf{u} = 0$ DUE TO UNIFORM MOTION~~

IN THE BEND

$$\mathbf{u} = (u_A, u_0) \text{ SUCH THAT } u_A = 0$$

$$u_0 = U \hat{e}_\theta$$

IF PARTICLE ROTATES

IN THE BEND, SPEED CANNOT BE UNIFORM.

$\Rightarrow$ PARTICLE RETAINS ORIENTATION

b) Acc. = $\frac{U^2}{R}$ WHERE ~~R IS BEND RADIUS~~

⑪

$$T(x, y) = 250 + 10xy$$

$$\frac{dT}{dt} = \frac{\partial T}{\partial x} x_t + \frac{\partial T}{\partial y} y_t = 10x_t x_t + \frac{10y_t}{10x_t y_t} 10x_t y_t$$

$$x_t = 2t, \quad y_t = 2$$

$$\Rightarrow \frac{dT}{dt} = 20t(2t) + 10(+^?)2$$

$$= 60t^2$$