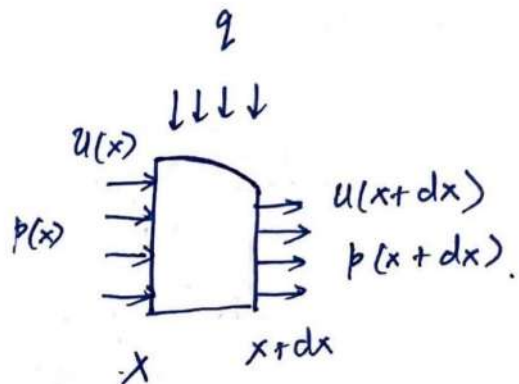
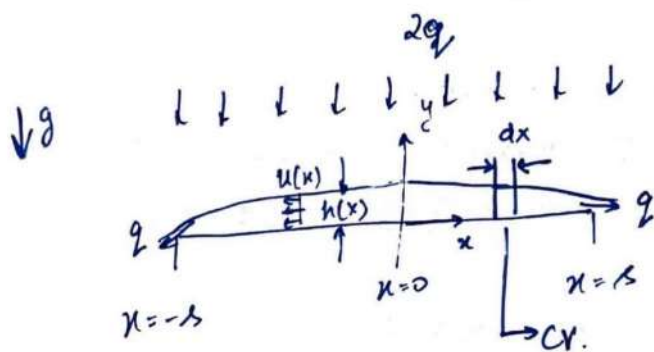


I
①

a) MASS CONSERVATION

$$- u(x) h(x) + u(x+dx) h(x+dx)$$

$$\Rightarrow \lim_{dx \rightarrow 0} \frac{u(x+dx) h(x+dx) - u(x) h(x)}{dx} = \left(\frac{q}{s} \right) dx$$

$$= \frac{q}{s} \Rightarrow \boxed{\frac{d}{dx}(uh) = \frac{q}{s}} \rightarrow (1)$$

LINEAR MOMENTUM (x)

$$- \rho u^2(x) h(x) + \rho u^2(x+dx) h(x+dx) = \int_0^{h(x)} p dy - \int_0^{h(x+dx)} p dy$$

$$= \frac{\rho g h^2(x)}{2} - \frac{\rho g h^2(x+dx)}{2}$$

DIVIDING BY dx

AND $\lim_{dx \rightarrow 0}$

$$\rho \frac{d}{dx}(u^2 h) = - \frac{\rho g}{2} \frac{d}{dx}(h^2) \quad \text{OR}$$

$$\boxed{\frac{d}{dx}(u^2 h) = - \frac{g}{2} \frac{d}{dx}(h^2)} \rightarrow (2)$$

→ WE ASSUMED
THAT PRESSURE
DUE TO MIST
($\sim \frac{\rho}{2} \left(\frac{q}{s} \right)^2$) IS SMALL

SOLVING (1) & (2).

$$uh = \frac{qx}{s} + C$$

AT $x=0$ $u=0$ (SYMMETRY)

$$\Rightarrow uh = \frac{qx}{s}$$

$$u^2 h = - \frac{gh^2}{2} + C$$

 $u=0$ AT $x=0$, $h=h_0$ (UNKNOWN)

$$\Rightarrow C = \frac{gh_0^2}{2}$$

$$\Rightarrow u^2 h = \frac{g}{2} (h_0^2 - h^2)$$

$$u = \frac{qx}{hs} \Rightarrow \frac{q^2 x^2}{hs^2} = \frac{g}{2} (h_0^2 - h^2)$$

$$\text{OR} \quad \frac{2q^2 x^2}{s^2} = -h^3 + h_0^3 h \quad \text{--- (2)}$$

TO FIND h_0 , CONSIDER BERNOLLI'S EQUATION BETWEEN $x=0$ & $x=s$

$$Pgh_0 = Pgh(s) + \frac{1}{2} \rho u(s)^2$$

$$u(s) = \frac{q}{h}$$

$$\Rightarrow \frac{q^2}{h^3(s)} = 2g(h_0 - h(s)) \Rightarrow q^2 - 2gh_0 h_s^2 - 2gh_s^3 \quad \text{--- (4)}$$

$$\text{FROM (2) AT } s \quad \frac{q^2}{h(s)} = \frac{g}{2} (h_0^2 - h^2(s))$$

$$q^2 = \frac{gh_0^2 h_s}{2} - \frac{gh_s^3}{2} \quad \text{--- (5)}$$

FROM (4) & (5).

$$\frac{h_0^2 h_s}{2} - \frac{h_s^3}{2} = 2h_0 h_s^2 - 2h_s^3 \Rightarrow h_s (3h_s^2 - 4h_s h_0 + h_0^2) = 0$$

$$\Rightarrow h_s = 0, h_0, \boxed{\frac{h_0}{3}}$$

SUBSTITUTING IN (5)

$$\Rightarrow \frac{3q^2}{h_0} = \frac{8}{9 \times 2} g h_0^2 \Rightarrow h_0 = 3 \left(\frac{q^2}{4g} \right)^{1/3}$$

FROM (3) IMPLICIT EQ^N IN $h(x)$

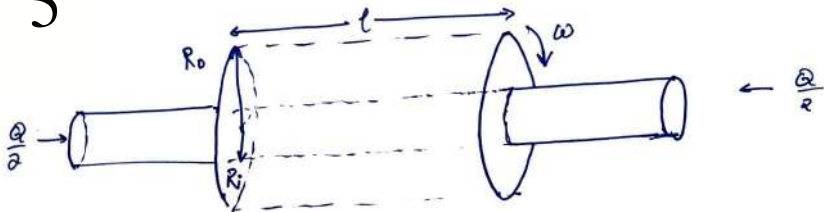
$$h^3(x) - h_0^3 h + \frac{2q^2 x^2}{s^2} = 0$$

b) $Pgh_0 = Pgh(x) + \frac{1}{2} \rho u^2(x)$, $P_s(x) = Pgh(x)$ (IGNORING p_0)

c) TOTAL FORCE $F(x) = \int_{-L}^L p(x) dx$ DOWNWARDS

$$= \rho g \frac{q}{s} \int_{-L}^L \frac{x}{u(x)} dx \quad (\text{Using (1)})$$

5



a) $\frac{1}{R} \frac{\partial}{\partial R} (R u_R) + \frac{1}{R} \frac{\partial u_\theta}{\partial \theta} = 0$ ASSUMING AXISYMMETRIC 2D

$$u_R = \frac{C}{R}, \quad \left| u_\theta(R_i) = 0, \quad u_\theta(R_o) = R_o \omega \right|$$

$$u_R(R_i) = \frac{Q}{2\pi R_i l}$$

b) $u_R = \frac{C}{R} \Rightarrow u_R = \frac{Q}{2\pi R l}$ TRANSPARATION AT R_o

$$u_R = \frac{Q}{2\pi R_o l}$$

c) R : SIMPLIFIED $\left[u_R \frac{du_R}{dR} - \frac{u_\theta^2}{R} = -\frac{1}{R} \frac{dp}{dR} \right]$

θ : $u_R \frac{du_\theta}{dR} + \frac{u_R u_\theta}{R} = \gamma \left(\frac{d^2 u_\theta}{dR^2} + \frac{1}{R} \frac{du_\theta}{dR} - \frac{u_\theta}{R^2} \right)$

SUBSTITUTING $u_R = \frac{Q}{2\pi R l} = \frac{C}{R}$

$$\frac{C}{R} \frac{du_\theta}{dR} + \frac{C u_\theta}{R^2} = \gamma \left(\frac{d^2 u_\theta}{dR^2} + \frac{1}{R} \frac{du_\theta}{dR} - \frac{u_\theta}{R^2} \right)$$

$$\Rightarrow \left[R^2 \frac{d^2 u_\theta}{dR^2} + R \left(1 - \frac{C}{\gamma} \right) \frac{du_\theta}{dR} - \left(1 + \frac{C}{\gamma} \right) u_\theta = 0 \right]$$

d) $\frac{C}{\gamma} = \frac{Q}{2\pi l \gamma} = \frac{1}{2}$

$$R^2 u_\theta'' + R \left(1 - \frac{1}{2} \right) u_\theta' - \left(1 + \frac{1}{2} \right) u_\theta = 0$$

LET $u_\theta = A R^n$

$$n(n-1) + \left(1 - \frac{\lambda}{2}\right)n - \left(1 + \frac{\lambda}{2}\right) = 0 \quad \text{OR} \quad n^2 - \frac{\lambda n}{2} + \left(\frac{\lambda}{2} + 1\right) = 0$$

$$n = -1, \frac{\lambda}{2} + 1$$

$$\Rightarrow U_0 = A n^{-1} + B n^{\frac{\lambda}{2} + 1}$$

SOLVE FOR A & B USING B.C.s

$$A = \omega R_0^2 \quad \begin{aligned} & \cancel{A R_i + B R_i^{\frac{\lambda}{2} - 1}} = 0 \\ & A R_0 + B R_0^{\frac{\lambda}{2} - 1} = \omega R_0 \end{aligned}$$

$$A = -B R_i^{\frac{\lambda}{2} - 2}$$

$$B R_0 R_i^{\frac{\lambda}{2} - 2} + B R_0^{\frac{\lambda}{2} - 1} = \omega R_0$$

$$B = \frac{\omega R_0^{-\lambda/2}}{1 - \left(\frac{R_i}{R_0}\right)^{\frac{\lambda}{2} + 2}}$$

$$U_0 = \frac{\omega R_0^{\frac{\lambda}{2} - 1}}{R_0^{\frac{\lambda}{2} - 2} - R_i^{\frac{\lambda}{2} - 2}} - \frac{\omega R_i^{\frac{\lambda}{2} - 2}}{R_0^{\frac{\lambda}{2} - 2} - R_i^{\frac{\lambda}{2} - 2}}$$

$$B = \frac{\omega R_0}{R_0^{\frac{\lambda}{2} - 1} - R_0 R_i^{\frac{\lambda}{2} - 2}}$$

$$= \frac{\omega}{R_0^{\frac{\lambda}{2} - 2} - R_i^{\frac{\lambda}{2} - 2}}$$

6

$$\underline{T} = 2\mu(\underline{\varepsilon}) \underline{\underline{\varepsilon}}, \quad u(\underline{\varepsilon}) = \mu_0 \left(\frac{1}{2} \underline{\underline{\varepsilon}} : \underline{\underline{\varepsilon}} \right)^{\frac{n-1}{2}}, \quad \underline{u} = (0, 0, w(r))$$

$$a) \quad S_{\theta\theta} = 0, \quad S_{\phi\phi} = S_{zz} = 0, \quad S_{\theta\phi} = S_{\phi\theta} = 0$$

$$S_{rz} = \frac{1}{2} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) = \frac{1}{2} \frac{dw}{dr}$$

$$S:S = S_{ij} S_{ij}$$

$$= S_{rz}^2 + S_{rz}^2$$

$$= 2 \left(\frac{1}{4} \left(\frac{dw}{dr} \right)^2 \right)$$

$$= \frac{1}{2} \left(\frac{dw}{dr} \right)^2$$

$$\mu(\underline{\underline{\varepsilon}}) = \mu_0 \left(\frac{1}{4} \left(\frac{dw}{dr} \right)^2 \right)^{\frac{n-1}{2}}$$

$$= \frac{\mu_0}{2} \left(\frac{1}{2} \frac{dw}{dr} \right)^{n-1}$$

$$b) \quad \frac{dp}{dz} = (\nabla \cdot \underline{T}) \cdot \hat{e}_z = \frac{1}{r} \frac{\partial}{\partial r} (r T_{rz})$$

$$T_{rz} = 2\mu S_{rz} = 2\mu_0 \left(\frac{1}{2} \frac{dw}{dr} \right)^{n-1} \left(\frac{1}{2} \frac{dw}{dr} \right)$$

$$= 2\mu_0 \left(\frac{1}{2} \frac{dw}{dr} \right)^n$$

$$\Rightarrow \frac{dp}{dz} = \frac{\mu_0}{2^{n-1}} \frac{1}{r} \frac{\partial}{\partial r} \left(r \left(\frac{dw}{dr} \right)^n \right)$$

$$\int 2^{n-1} \frac{db}{dz} r dr = \mu_0 \int \frac{d}{dr} \left(r \left(\frac{du_z}{dr} \right)^n \right) dr$$

$$\left. \frac{du_z}{dr} \right|_{r=0} = 0$$

$$2^{n-1} \frac{db}{dz} \frac{r^2}{2} + C = \mu_0 \left(r \left(\frac{du_z}{dr} \right)^n \right)$$

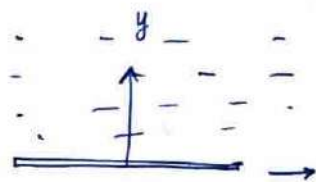
$$\Rightarrow r \left(\frac{du_z}{dr} \right)^n = \frac{2^{n-2}}{\mu_0} \frac{db}{dz} r^2 \Rightarrow \frac{du_z}{dr} = \left(\frac{2^{n-2}}{\mu_0} \frac{db}{dz} \right)^{\frac{1}{n}} r^{\frac{1}{n}}$$

$$\Rightarrow u_z = C + \left(\frac{2^{n-2}}{\mu_0} \frac{db}{dz} \right)^{\frac{1}{n}} \frac{r^{1+\frac{1}{n}}}{1+\frac{1}{n}}$$

$$u_z(R) = 0 \quad (\text{NO-SLIP})$$

$$\Rightarrow C = - \left(\frac{2^{n-2}}{\mu_0} \frac{db}{dz} \right)^{\frac{1}{n}} \frac{n R^{\frac{1+n}{n}}}{1+n}$$

$$\Rightarrow u_z(r) = - \left(\frac{2^{n-2}}{\mu_0} \frac{db}{dz} \right)^{\frac{1}{n}} \frac{n}{1+n} \left(R^{\frac{1+n}{n}} - r^{\frac{1+n}{n}} \right)$$



$$u(t) = A\sqrt{\nu t} H(t)$$

GOV. EQUATION

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$$

$$u(y \rightarrow \infty, t) = 0, \quad u(y=0, t) = A\sqrt{\nu t} \quad (\text{for } t > 0, H(t) = 1)$$

$$u(y, 0) = 0 \quad (\text{QUIESCENT INITIAL CONDITION})$$

$$\text{LET } u = B(\nu t)^\alpha f(\eta), \quad \eta = \frac{y}{2\sqrt{\nu t}}, \quad \frac{\partial \eta}{\partial y} = \frac{1}{2\sqrt{\nu t}}, \quad \frac{\partial \eta}{\partial t} = \frac{y}{2\sqrt{\nu}} \left(-\frac{1}{2t^{3/2}} \right)$$

$$\frac{\partial u}{\partial t} = B\nu^\alpha t^{\alpha-1} f + B(\nu t)^\alpha f' \left(-\frac{\eta}{2t} \right) = -\frac{\eta}{2t}$$

$$= B(\nu t)^\alpha \left(\frac{\alpha f}{t} - \frac{f' \eta}{2t} \right)$$

$$\frac{\partial u}{\partial y} = \frac{B(\nu t)^\alpha f'}{2\sqrt{\nu t}}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{B(\nu t)^\alpha f''}{2(2\sqrt{\nu t})}$$

$$\Rightarrow B(\nu t)^\alpha \left(\frac{\alpha f}{t} - \frac{f' \eta}{2t} \right) = \nu \frac{B(\nu t)^\alpha f''}{4\sqrt{t}}$$

$$\boxed{\alpha f - \frac{f' \eta}{2} = \frac{f''}{4}} \quad \text{--- DA } u(y=0, t) = A\sqrt{\nu t} = B(\nu t)^\alpha f(0)$$

$$\Rightarrow \boxed{f'' + 2f' \eta = 2f} \quad \text{--- ①}$$

$$\text{B.C. } \boxed{f(0) = \frac{A}{B}, \quad f(\eta \rightarrow \infty) = 0}$$

$$\Rightarrow \alpha = \frac{1}{2} \quad \boxed{B f(0) = A}$$

TO SOLVE ①, WE NOTE THAT $f(\eta) = \eta$ IS A SOLUTION

$$\text{LET } f(\eta) = \eta g(\eta), \quad f' = \eta g' + g, \quad f'' = \cancel{g'' + 2g'} + \eta g'' + 2g'$$

$$\eta g'' + 2g' + 2(\eta g' + g)\eta = 2\eta g \quad \Rightarrow \quad \eta g'' + 2g' + 2\eta^2 g' = 0$$

$$\Rightarrow g'' = -\left(\frac{2}{\eta} + 2\eta\right)g'$$

$$\frac{g''}{g'} = \frac{-2}{\eta} - 2\eta \Rightarrow \ln g' = -\ln \eta^2 - \eta^2 + \ln K$$

$$\ln \left(\frac{g \eta^2}{K} \right) = -\eta^2 \Rightarrow g' = \frac{K}{\eta^2} e^{-\eta^2}$$

$$= g = K \int \frac{1}{\eta^2} e^{-\eta^2} d\eta + C$$

$$\int \frac{1}{\eta^2} e^{-\eta^2} d\eta = -\frac{1}{\eta} e^{-\eta^2} + 2 \int \eta e^{-\eta^2} \left(-\frac{1}{\eta}\right) d\eta$$

$$= -\frac{1}{\eta} e^{-\eta^2} - 2 \int e^{-\eta^2} d\eta$$

$$\Rightarrow f = C_1 - K \left[e^{-\eta^2} + 2\eta \int e^{-\eta^2} d\eta \right]$$

$$f(\eta=0) = -K = \frac{A}{B}$$

$$\Rightarrow f = C_1 + \frac{A}{B} \left[e^{-\eta^2} + 2\eta \int_0^\eta e^{-s^2} ds \right]$$

$$f(\eta \rightarrow \infty) = 0 \Rightarrow \lim_{\eta \rightarrow \infty} \left[C_1 + 2\frac{A}{B} \int_0^\infty e^{-s^2} ds \right] = 0 \Rightarrow C_1 = -\frac{A\sqrt{\pi}}{B}$$

$$\Rightarrow f = \frac{A}{B} \left[e^{-\eta^2} + 2\eta \int_0^\eta e^{-s^2} ds - \eta\sqrt{\pi} \right]$$

$$u(y) = B \sqrt{\gamma t} f(\eta) = A (\gamma t)^{\frac{1}{2}} \left[e^{-\eta^2} + \sqrt{\pi} \operatorname{erf}(\eta) - \eta\sqrt{\pi} \right]$$

$$\left(\eta = \frac{y}{2\sqrt{\gamma t}} \right)$$

8

FOR AN IDEAL VORTEX LINE

$$u_0 = \int \frac{\gamma}{2\pi h}$$

FOR DECAYING, $u_0 \rightarrow u_0(h, t)$

Θ: MOMENTUM EQⁿ

$$\frac{\partial u_0}{\partial t} = \gamma \left(\frac{1}{h} \frac{\partial}{\partial t} \left(h \frac{\partial u_0}{\partial h} \right) - \frac{u_0}{h^2} \right)$$

$$\frac{\partial u_0}{\partial t} = \gamma \left(\frac{1}{h} \frac{\partial u_0}{\partial h} + \frac{\partial^2 u_0}{\partial h^2} - \frac{u_0}{h^2} \right)$$

$$u_0(x=0, t) = 0, \quad \underline{u(x, t-)} \Rightarrow f(\eta=0) = 0$$

$$u_0(x \rightarrow \infty, t) = \frac{1}{2\pi h} \Rightarrow f(\eta \rightarrow \infty) = 1$$

$$u_0(x, t=0) = \frac{1}{2\pi h} \Rightarrow f(\eta \rightarrow \infty) = 1$$

$$f'' = \left(\frac{1}{\eta} - \frac{\eta}{2}\right) f' \Rightarrow \ln f' = \ln \eta - \frac{\eta^2}{4} + \ln K$$

$$\Rightarrow \frac{f'}{\eta K} = e^{-\eta^2/4} \Rightarrow f' = K \eta e^{-\eta^2/4}$$

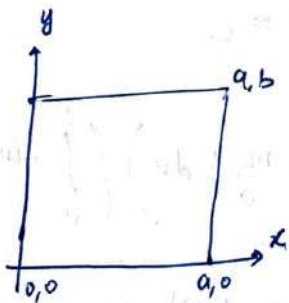
$$f = K \int \eta e^{-\eta^2/4} d\eta + C$$

$$f(\eta=0) = 0 \Rightarrow f(\eta) = K \int_0^\eta t e^{-t^2/4} dt + C = \frac{K}{2} \int_0^{\eta^2} e^{-t/4} dt$$

$$\Rightarrow f(\eta) = \frac{4K}{2} (1 - e^{-\eta^2/4}) = -\frac{4K}{2} (e^{-\eta^2/4} - 1)$$

$$f(\eta \rightarrow \infty) = 1 \Rightarrow \frac{4K}{2} = 1 \Rightarrow f(\eta) = 1 - e^{-\eta^2/4}$$

$$u_0 = \frac{1}{2\pi} \left(1 - e^{-\frac{h^2}{4\pi t}}\right)$$



$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{\mu} \frac{dp}{dz} = \text{const.}$$

FOR HOMOGENEOUS EQN

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

$$w(0, y) = 0$$

$$w(x, 0) = 0$$

$$w = X(x)Y(y)$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \Rightarrow \frac{Y''}{Y} = \lambda, \quad \frac{X''}{X} = -\lambda$$

$$X = A e^{i\sqrt{\lambda}x} + B e^{-i\sqrt{\lambda}x}$$

$$X(0) = 0 \Rightarrow A = B, \quad X(a) = 0 \Rightarrow 2iA \sin(\sqrt{\lambda}a) = 0$$

$$\Rightarrow \lambda_n = \left(\frac{n\pi}{a}\right)^2$$

→ EIGEN FDX
VALUES X

$$W(x,y) = \frac{4C}{\pi^4} \sum_m \sum_n \frac{1}{mn} \frac{1}{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} (1-(-1)^m) (1-(-1)^n) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

SINCE ONLY ODD m & n SURVIVE

AND $C = \frac{1}{\mu} \frac{db}{dx}$

$$\Rightarrow W(x,y) = \frac{4}{\mu\pi^4} \frac{db}{dx} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(2m+1)(2n+1)} \frac{4}{\left(\frac{2m+1}{a}\right)^2 + \left(\frac{2n+1}{b}\right)^2} \sin\left(\frac{(2m+1)\pi x}{a}\right) \sin\left(\frac{(2n+1)\pi y}{b}\right)$$

10

FOR $f(x,y) = \begin{cases} c & x+y < a \\ -c & x+y > a \end{cases}$

WE CAN DETERMINE FOURIER COEFFICIENTS A_{mn} AS

$$f(x,y) = \sum_m \sum_n A_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad \boxed{b=a}$$

$$\iint f(x,y) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dx dy = \frac{4}{ab} A_{mn}$$

$$\Rightarrow A_{mn} = \frac{4}{ab} \left[\int_{x+y < a} c \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) ds - \int_{x+y > a} c \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) ds \right]$$

$$= \frac{4c}{ab} \left[\int_{x+y < a} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) ds - \int_{x+y > a} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) ds \right]$$

$\text{I} \qquad \qquad \qquad \text{II}$

$\text{I} \rightarrow \int_0^a \int_0^{a-x} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dy dx$

$$= \int_0^a \sin\left(\frac{m\pi x}{a}\right) \left[-\cos\left(\frac{n\pi y}{b}\right) \frac{b}{n\pi} \right]_0^{a-x} dx$$

$$\begin{aligned}
I &= \frac{a}{n\pi} \int_0^a \sin \frac{m\pi x}{a} \left(1 - \cos \frac{n\pi}{a}(a-x) \right) dx \\
&= \frac{a}{n\pi} \left[\int_0^a \sin \frac{m\pi x}{a} dx - \cos n\pi \int_0^a \sin \frac{m\pi x}{a} \cos \frac{n\pi x}{a} dx \right] \\
&= \frac{a}{n\pi} \left[\frac{a}{m\pi} (1 - \cos(m\pi)) - \frac{\cos(n\pi)}{2} \left[\frac{1 - \cos(m+n)\pi}{(m+n)\pi} + \frac{1 - \cos(m-n)\pi}{(m-n)\pi} \right] \right] \\
&= \frac{a^2}{n\pi^2} \left(\frac{1 - \cos m\pi}{m} - \frac{\cos n\pi}{2} \left[\frac{2m - 2m \cos(m\pi) \cos(n\pi)}{m^2 - n^2} \right] \right) \\
&= \frac{a^2}{n\pi^2} \left(\frac{m^2 (1 - \cos(n\pi)) + n^2 (1 - \cos(m\pi))}{m(m^2 - n^2)} \right) \\
&= \frac{a^2}{mn\pi^2(m^2 - n^2)} \left[m^2 (1 - \cos(n\pi)) + n^2 (1 - \cos(m\pi)) \right]
\end{aligned}$$

SIMILARLY

$$\begin{aligned}
II &= \frac{a^2}{mn\pi^2(m^2 - n^2)} \left[(m^2 - n^2) \cos(m\pi) \cos(n\pi) + n^2 \cos n\pi - m^2 \cos m\pi \right] \\
&\quad \text{(VERIFY)}
\end{aligned}$$

$$\Rightarrow A_{mn} = \frac{4C}{mn\pi^2(m^2 - n^2)} \left[m^2 (1 - \cos n\pi) + n^2 (1 - \cos m\pi) - (m^2 - n^2) \cos m\pi \cos n\pi + m^2 \cos m\pi - n^2 \cos n\pi \right]$$

ASSUMING $W(x, y) = \sum_m \sum_n W_{mn} \sin \left(\frac{m\pi x}{a} \right) \sin \left(\frac{n\pi y}{a} \right)$

$$\nabla^2 W = \sum_{m,n} \left(\frac{m^2 + n^2}{a^2} \right) W_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a}$$

$$\Rightarrow \left(\frac{m^2 + n^2}{a^2} \right) W_{mn} = A_{mn} \rightarrow \text{SOLVE FOR } W_{mn}$$

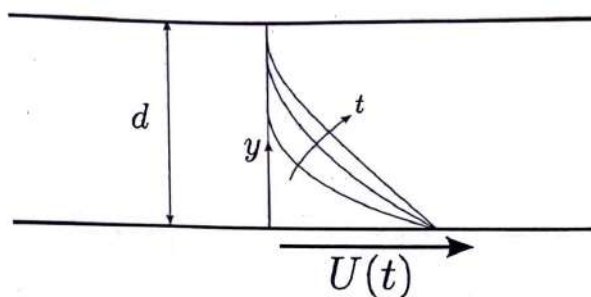


Figure 2: Start-up of shear flow between two plates.

3. Let us consider the start-up of the flow of fluid of viscosity μ between two walls separated by distance d and caused by a sudden impulsive movement of the bottom wall in the shown figure. The top wall is fixed and the fluid is initially at rest. The bottom wall velocity $U(t)$ is such that $U(t) = 0$ for $t \leq 0$ and $U(t) = U_0$ for $t > 0$. (25)
- State all the boundary and initial conditions. (3)
 - Using suitable assumptions derive the PDE governing this unsteady flow in the dimensionless form. Clearly state the all the scales used for dimensionless variables. (12)
 - Solve the above equation using variable separation. (7)
 - Derive the later-time $t \rightarrow \infty$ limit of the solution. Compare it to the steady state Couette-flow solution. (3)

a) $u(y=0, t) = U$ $u(y, 0) = 0$
 $u(d, t) = 0$

b) UNIDIRECTIONAL, 1D FLOW DRIVEN BY WALLS

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} \right)$$

$$\Rightarrow \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \quad \cdot \quad \tilde{u} = \frac{u}{U_0}, \quad \tilde{t} = \frac{\nu t}{d^2}, \quad \tilde{y} = \frac{y}{d}$$

SINCE THE FLOW DIFFUSES UP $[t] = \frac{\nu t}{d^2}$

$$\frac{\nu U_0}{d^2} \frac{\partial \tilde{u}}{\partial \tilde{t}} = \frac{\nu U_0}{d^2} \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} \quad \text{OR} \quad \frac{\partial \tilde{u}}{\partial \tilde{t}} = \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} \quad \begin{aligned} \tilde{u}(\tilde{y}=0, \tilde{t}) &= 1 \\ \tilde{u}(\tilde{y}=1, \tilde{t}) &= 0 \\ \tilde{u}(\tilde{y}, 0) &= 0 \end{aligned}$$

c) LET $V = u - (1-y)$ (DROPPING $\tilde{c}$)

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} \quad \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y^2}$$

$$v(y, 0) = -(1-y) \rightarrow \text{I.C.}$$

$$v(0, t) = 0 \rightarrow \text{B.C.s}$$

$$v(1, t) = 0$$

LET $V = T(t) Y(y)$

$$T'Y = TY'' \quad \text{OR} \quad \frac{T'}{T} = \frac{Y''}{Y} = \lambda$$

$$T = Ae^{\lambda t}$$

$$Y = C_1 e^{\sqrt{\lambda} y} + C_2 e^{-\sqrt{\lambda} y}$$

$$Y(0) = 0 \Rightarrow C_2 = -C_1$$

$$Y(1) = 0 \Rightarrow C_1 \sinh \sqrt{\lambda} = 0 \quad e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}$$

FOR $\lambda > 0$ ONLY $\lambda = 0$ (TRIVIAL SOLUTION)
FOR $\lambda < 0$ OR $\lambda = -k^2$

$$Y(1) = 0 \Rightarrow C_1 \sin ky = 0$$

$$\Rightarrow k = \pm n\pi \quad n = 1, 2, \dots$$

$$\Rightarrow u = \sum_{n=1}^{\infty} A_n e^{-n^2 \pi^2 t} \sin(n\pi y)$$

$$v(t=0) = -(1-y) = \sum_{n=1}^{\infty} A_n \sin(n\pi y)$$

$$-\int_0^1 (1-y) \sin m\pi y dy = \sum_{n=1}^{\infty} A_n \int_0^1 \sin(n\pi y) \sin(m\pi y) dy$$

$$A_m = \frac{\int_0^1 (y-1) \sin m\pi y dy}{\int_0^1 \sin^2 m\pi y dy} = \frac{\frac{y}{m\pi} (-\cos m\pi y) \Big|_0^1 + \frac{1}{m\pi} \int_0^1 \cos m\pi y dy}{\frac{1}{2}}$$

$$A_m = -\frac{2}{m\pi}$$

$$\Rightarrow v = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{e^{-n^2 \pi^2 t}}{n} \sin(n\pi y)$$

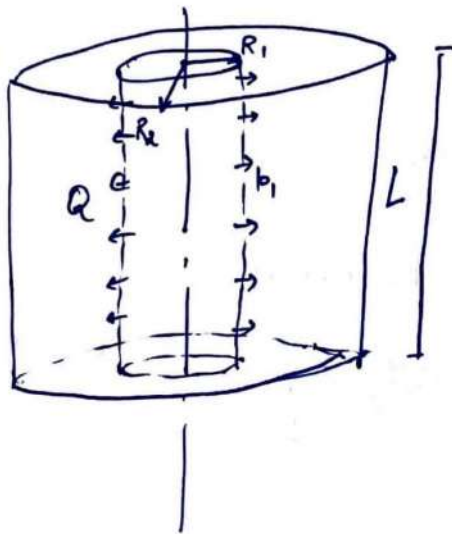
$$u = 1 - y - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{e^{-n^2 \pi^2 t}}{n} \sin(n\pi y)$$

DIMENSIONAL

$$\frac{u}{U_0} = \left(1 - \frac{y}{d}\right) - \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-\frac{n^2 \pi^2 \gamma t}{d^2}} \sin\left(n\pi \frac{y}{d}\right)$$

d) For $t \rightarrow \infty$

$$\frac{u}{U_0} = \left(1 - \frac{y}{d}\right) = U_0 \left(1 - \frac{y}{d}\right) \rightarrow \text{COUETTE FLOW SOLUTION}$$



ASSUMING AXISYMMETRIC FLOW

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_r}{\partial r} \right) + \frac{\partial u_z}{\partial z} = 0$$

ID (RADIAL) FLOW.

$$\Rightarrow \frac{1}{r} \frac{\partial}{\partial r} \left(r u_r \right) = 0 \Rightarrow u_r = \frac{C}{r}$$

$$\oint u_r (2\pi R_1) L = Q \Rightarrow \boxed{C = \frac{Q}{2\pi L}}$$

μ : MOMENTUM (STEADY)

$$u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} = -\frac{1}{\rho} \frac{dp}{dr} + \nu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_r}{\partial r} \right) - \frac{u_r}{r^2} \right)$$

$$u_r \frac{\partial u_r}{\partial r} + \frac{1}{\rho} \frac{dp}{dr} = \frac{\nu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_r}{\partial r} \right) - \frac{u_r}{r^2}$$

$$u_r = \frac{C}{r}, \quad \frac{\partial u_r}{\partial r} = -\frac{C}{r^2}$$

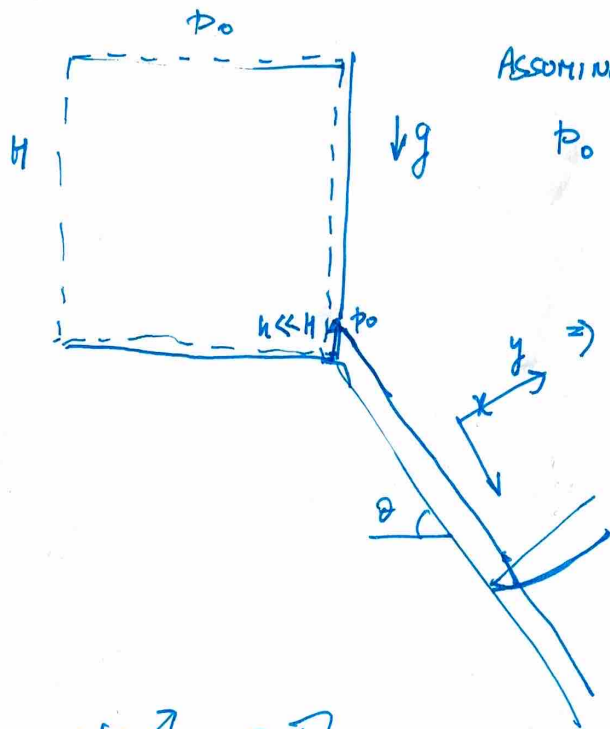
$$-\frac{C^2}{r^3} + \frac{1}{\rho} \frac{dp}{dr} = \frac{\nu}{r} \frac{d}{dr} \left(-\frac{C}{r} \right) - \frac{\nu C}{r^3}$$

$$\frac{1}{\rho} \frac{dp}{dr} = \frac{\nu C}{r^3} - \frac{\nu C}{r^3} + \frac{C^2}{r^3} \Rightarrow \frac{dp}{dr} = \frac{\rho C^2}{r^3}$$

$$\Rightarrow \int_{p_1}^p dp = \rho C^2 \int_{R_1}^r \frac{dr}{r^3} \Rightarrow p - p_1 = \rho C^2 \left(\frac{1}{R_1^2} - \frac{1}{r^2} \right)$$

$$p = p_1 + \rho \left(\frac{Q}{2\pi L} \right)^2 \left(\frac{1}{R_1^2} - \frac{1}{r^2} \right)$$

Q2



ASSUMING INVISCID FLOW IN TANK

$$p_0 + \frac{1}{2} \rho v^2 = p_0 + \rho g H$$

$$\Rightarrow v = \sqrt{2gH}$$

$$\Rightarrow \text{FLOW RATE (PER UNIT DEPTH)} = h\sqrt{2gH}$$

SIMPLIFYING N.S. EQUATIONS

• FULLY DEVELOPED

• STEADY

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow v = \text{const.}$$

$$v = 0 \text{ AT } y = 0$$

$$\Rightarrow v = 0.$$

$$x: \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \rho g \sin \theta$$

$$y: \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \rho g \cos \theta$$

$$\Rightarrow \frac{\partial p}{\partial y} = -\rho g \cos \theta, \quad \frac{\partial p}{\partial x} = \mu \frac{\partial^2 v}{\partial y^2} + \rho g \sin \theta$$

$$\text{AT } y = 0 \quad (u = 0)$$

$$\text{AT } y = d \quad \left(\mu \frac{\partial v}{\partial y} \Big|_{\text{water}} = \mu \frac{\partial v}{\partial y} \Big|_{\text{air}} \right) \Rightarrow \frac{\partial v}{\partial y} \approx 0$$

$$p_{\text{water}} = p_{\text{air}}$$

ATMOSPHERIC

Since pressure is uniform $p_{air} = p_0 = \text{const.}$

$$\Rightarrow p_{water} = p_0 \text{ AT } y = \delta \quad \forall x \Rightarrow \left. \frac{\partial p}{\partial x} \right|_{y=\delta} = 0$$

$$\frac{\partial p}{\partial y} = -\rho g \cos \theta \Rightarrow p = -\rho g \cos \theta y + C(x)$$

$$p(\delta) = p_0 = C - \rho g \cos \theta \delta.$$

$$\Rightarrow p = p_0 + \rho g \cos \theta (\delta - y) \Rightarrow C(x) = p_0 + \rho g \cos \theta \delta = \text{const.}$$

$$\Rightarrow \frac{\partial p}{\partial x} = 0 \Rightarrow \mu \frac{\partial^2 u}{\partial y^2} = -\rho g \sin \theta$$

$$\Rightarrow u = -\frac{\rho g \sin \theta}{2\mu} y^2 + C_1 y + C_2$$

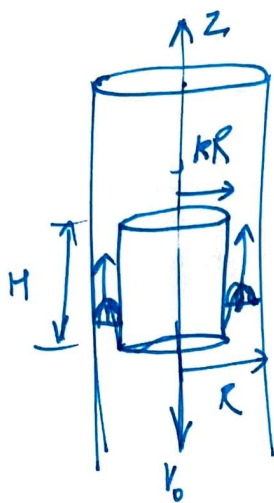
$$u(y=0) = 0 \Rightarrow C_2 = 0, \quad \frac{\partial u}{\partial y} = -\frac{\rho g \sin \theta}{\mu} y + C_1$$

$$\left. \frac{\partial u}{\partial y} \right|_{\delta} = 0 \Rightarrow C_1 = \frac{\rho g \sin \theta}{\mu} \delta \Rightarrow u = \frac{\rho g \sin \theta}{\mu} \left(\delta y - \frac{y^2}{2} \right)$$

$$\int_0^{\delta} u dy = \frac{\rho g \sin \theta}{\mu} \left(\frac{\delta^3}{2} - \frac{\delta^3}{6} \right) = \frac{\rho g \sin \theta \delta^3}{3\mu} = h \sqrt{2gH}$$

(FIND δ)

(4)



IN CYLINDRICAL COORDINATES
• ASSUME AXISYMMETRIC FLOW

• $u_\theta = 0$

• FULLY DEVELOPED OVER SLUG HEIGHT $\Rightarrow \frac{\partial}{\partial z} = 0$

• STEADY

$$\frac{\partial}{\partial \theta} = 0$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

$$\Rightarrow r u_r = \text{const.}$$

$$u_r(r=kR) = 0$$

$$\Rightarrow u_r = 0$$

$$\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} \right)$$

$$\Rightarrow \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) = \frac{1}{\mu} \frac{\partial p}{\partial z}$$

FROM θ : & r : MOMENTUM EQUATIONS

$$\frac{\partial p}{\partial r} = 0, \quad \frac{\partial p}{\partial \theta} = 0$$

$$\Rightarrow \frac{\partial p}{\partial z} = \frac{dp}{dz} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) = \text{const.}$$

(SHOW THIS IN EXAMS!)

$$\Rightarrow \frac{dp}{dz} \frac{r^2}{2} + C_1 = r \frac{\partial u_z}{\partial r} \Rightarrow u_z = \frac{dp}{dz} \frac{r^2}{4} + C_1 \ln r + C_2$$

BOUNDARY CONDITIONS

$$u_z(r=kR) = -V_0, \quad u_z(r=R) = 0$$

$$\frac{dp}{dz} (kR)^2 + C_1 \ln(kR) + C_2 = -V_0 \Rightarrow V_0 = C_1 (\ln R - \ln kR) + \frac{dp}{dz} (R^2 - k^2 R^2)$$

$$\frac{dp}{dz} R^2 + C_1 \ln R + C_2 = 0$$

$$\Rightarrow C_1 = \frac{V_0}{\ln R - \ln kR} - \frac{dp}{dz} \frac{R^2(1-k^2)}{\ln R - \ln kR}$$

$$C_2 = -\frac{dp}{dz} R^2 - C_1 \ln R = -\frac{dp}{dz} R^2 + \frac{dp}{dz} \frac{R^2(1-k^2)}{\ln \frac{1}{k}} \ln R - \frac{V_0 \ln R}{\ln(\frac{1}{k})}$$

$$\Rightarrow C_2 = \frac{dp}{dz} \frac{R^2}{\ln \frac{1}{k}} \left(\frac{1-k^2}{\ln R} - 1 \right) + \frac{V_0 \ln(R)}{\ln(k)}$$

BUT $\frac{dp}{dz}$ IS NOT KNOWN. $\frac{dp}{dz}$ CAN BE OBTAINED USING MASS CONSERVATION.

For 2D flows, if vorticity field is given $\underline{\omega}$, the velocity field is given by,

$$\underline{u}(\underline{x}) = \frac{1}{2\pi} \nabla \times \left(\int_{\mathbb{R}^2} \ln |\underline{x} - \underline{y}| \underline{\omega}(\underline{y}) d\underline{y} \right), \quad (1)$$

where $\underline{y}$ and $\underline{x}$ are coordinate vectors in 2D space $\mathbb{R}^2$ and ∇ denotes gradient w.r.t $\underline{x}$. Simplify the above expression using index notation. **(5)**

Answer:-

$$u_i = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\epsilon_{ijk}}{|\underline{x} - \underline{y}|} \frac{\partial}{\partial x_j} (|\underline{x} - \underline{y}|) d\underline{y} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\epsilon_{ijk} r_j}{r^2} \omega_k d\underline{y}$$

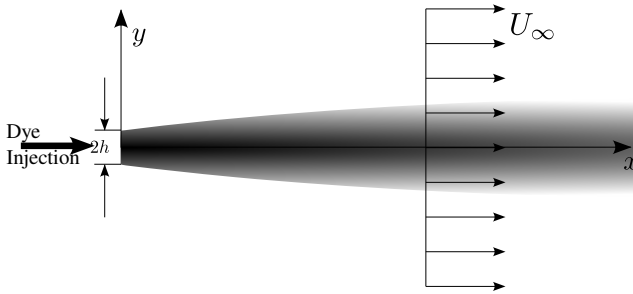
where $\underline{r} = \underline{x} - \underline{y}$ and $r = |\underline{r}|$. In the vector notation,

$$\underline{u} = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{\underline{r} \times \underline{\omega}}{r^2} d\underline{y}.$$

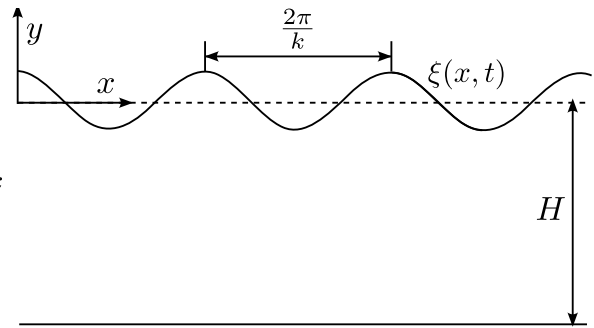
A dye in a flow can be modeled as a passive scalar with concentration field $c(\underline{x}, t)$ (intensive quantity). For a passive scalar with diffusivity coefficient D , the governing equation is given by,

$$\frac{\partial c}{\partial t} + \underline{u} \cdot \nabla c = D \nabla^2 c. \quad (2)$$

Consider a dye being injected very slowly at $x = 0$ in a uniform, steady, parallel, and 2D flow as shown below. The dye concentration is c_0 within $-h \leq y \leq h$ at $x = 0$ at all times. As the dye gets convected downstream in x , it spreads in y as shown. The value of concentration within $-h \leq y \leq h$ also changes.



Problem 2



Problem 3

- (a) For steady state conditions in the uniform parallel flow as shown, simplify the passive scalar equation. Ignore $\frac{\partial^2}{\partial x^2}$ term compared to $\frac{\partial^2}{\partial y^2}$ **(2)**

Answer:- For steady state and ignoring the x diffusion term,

$$U_{\infty} \frac{\partial c}{\partial x} = D \frac{\partial^2 c}{\partial y^2}$$

(b) Clearly specify all the boundary conditions on c . **(3)**

Answer:- $c(x, y \rightarrow \pm\infty) = 0$ and $c(x \rightarrow 0, y) = c_0$ for $|y| < h$ and $c(x \rightarrow 0, y) = 0$ for $|y| > h$.

(c) Solve the simplified PDE from part 2a. Consider Fourier-transform in y such that,

$$\hat{c}(k, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(y, x) e^{-iky} dy, \quad c(y, x) = \int_{-\infty}^{\infty} \hat{c}(k, x) e^{iky} dk. \quad (3)$$

Using this, derive a simple first order PDE for $\hat{c}(k, y)$ from the simplified PDE. Solve it to obtain the solution $c(x, y)$. **(15)**

Answer:- Taking the Fourier transform,

$$\frac{\partial \hat{c}}{\partial x} = -\frac{D}{U_{\infty}} k^2 \hat{c} \implies \hat{c}(k, x) = g(k) e^{-\frac{Dk^2 x}{U_{\infty}}}.$$

Since $c(y, x \rightarrow 0)$ is known, we can explicitly evaluate $g(k)$. To simplify calculation, we use,

$$c(y, x) = \int_{-\infty}^{\infty} g(k) \exp \left[-\frac{Dxk^2}{U_{\infty}} + ik y \right] dk.$$

Using

$$g(k) = \hat{c}(k, x \rightarrow 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(z, x \rightarrow 0) e^{-ikz} dz,$$

we get,

$$c(y, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(z, x \rightarrow 0) e^{-ikz} dx \exp \left[-\frac{Dxk^2}{U_{\infty}} + ik y \right] dk,$$

which can be simplified as,

$$\begin{aligned} c(y, x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} c(z, x \rightarrow 0) \int_{-\infty}^{\infty} \exp \left[-\frac{Dxk^2}{U_{\infty}} + ik(y - z) \right] dk dz \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} c(z, x \rightarrow 0) \exp \left[-\frac{U_{\infty}(y - z)^2}{4Dx} \right] \int_{-\infty}^{\infty} \exp \left(-\left(k \sqrt{\frac{Dx}{U_{\infty}}} - i \frac{y - z}{2} \sqrt{\frac{U_{\infty}}{Dx}} \right)^2 \right) dk dz, \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} c(z, x \rightarrow 0) \exp \left[-\frac{U_{\infty}(y - z)^2}{4Dx} \right] \sqrt{\frac{U_{\infty}\pi}{Dx}} dz, \\ &= \frac{c_0}{2} \sqrt{\frac{U_{\infty}}{\pi Dx}} \int_{-h}^h \exp \left[-\frac{U_{\infty}(y - z)^2}{4Dx} \right] dz. \end{aligned}$$

Alternatively, one can explicitly calculate,

$$\begin{aligned} g(k) &= \hat{c}(k, x \rightarrow 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(y, x \rightarrow 0) e^{-iky} dy = \frac{1}{2\pi} \int_{-h}^h c_0 e^{-iky} dy = \frac{c_0}{2\pi ik} (e^{-ikh} - e^{ikh}) \\ &= -\frac{c_0}{\pi} \frac{\sin(kh)}{k}. \end{aligned}$$

Substituting to obtain the concentration,

$$c(y, x) = -\frac{c_0}{\pi} \int_{-\infty}^{\infty} \frac{\sin(kh)}{k} \exp \left[-\frac{Dxk^2}{U_{\infty}} + ik y \right] dk. \quad (4)$$

3. An infinitely long cylinder of radius R is immersed in a Newtonian fluid. The fluid is motionless. The cylinder begins to rotate about its axis at $t = 0$ with angular velocity $\Omega \hat{e}_z$.



Figure 2: Infinitely long rotating cylinder

- (a) Assuming axi-symmetric flow, calculate the velocity distribution at steady state by simplifying the Navier-Stokes equations in cylindrical coordinates. **(15)**

Answer:- For axi-symmetric flow u_r and u_θ do not vary in θ . From continuity equation and $u_r(R) = 0$,

$$\frac{1}{r} \frac{\partial(r u_r)}{\partial r} = 0 \implies u_r = 0. \quad (13)$$

From θ momentum equation (at steady state),

$$\frac{d^2 u_\theta}{dr^2} + \frac{1}{r} \frac{du_\theta}{dr} - \frac{u_\theta}{r^2} = 0. \quad (14)$$

Taking $u_\theta = r^n$, we get $n = 1, -1$. For $r \rightarrow \infty$, flow velocity must be finite $\implies u_\theta = \frac{C}{r}$.

At $r = R$, $u_\theta = \Omega R$, we get $u_\theta = \frac{\Omega R^2}{r}$. From r momentum equation,

$$-\frac{u_\theta^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r}. \quad (15)$$

- (b) Consider the start-up flow at initial times due to cylinder rotation. In the small radius limit $R \rightarrow 0$ and $\Omega R^2 \rightarrow \frac{\Gamma}{2\pi}$ (a constant), simplify the governing equations and the boundary conditions. **(15)**

Answer:- Since, for $\Omega R^2 \rightarrow \frac{\Gamma}{2\pi}$ as $R \rightarrow 0$, we get the boundary conditions as,

$\lim_{R \rightarrow 0} R u_\theta(r = R \rightarrow 0, t) = \frac{\Gamma}{2\pi}$, $u_\theta(r \rightarrow \infty, t) = 0$. Boundary conditions for u_r remain the same. For $p(r \rightarrow \infty, t) = p_\infty$. The continuity equation still yields $u_r = 0$. The r equation also remains same. For θ momentum equation, we obtain,

$$\frac{\partial u_\theta}{\partial t} = \nu \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} \quad (16)$$

- (c) Using a similarity variable $\eta = \frac{r}{\sqrt{\nu t}}$ solve the simplified equations obtained in part 3b to obtain the unsteady velocity field. Confirm that for $t \rightarrow \infty$ this field approaches the solution in part 3a. **(15)**

Answer:- Looking at the steady state solution, we take $u_\theta = \frac{\Gamma}{2\pi r} f(\eta)$ as the dependent variable. Substituting in equation 16, we obtain,

$$\frac{-f'}{2t\sqrt{\nu t}} = \nu \left[\frac{1}{r} \frac{f'}{r\sqrt{\nu t}} - \frac{f}{r^2} + \frac{f''}{r\nu t} - \frac{f'}{r^2\sqrt{\nu t}} + \frac{2f}{r^3} - \frac{f'}{r^2\sqrt{\nu t}} - \frac{f}{r^3} \right], \quad (17)$$

which yields,

$$\frac{-f'}{2t\sqrt{\nu t}} = \nu \left[\frac{f''}{r\nu t} - \frac{f'}{r^2\sqrt{\nu t}} \right] = \frac{f''}{rt} - \frac{f'\sqrt{\nu}}{r^2\sqrt{t}}. \quad (18)$$

Using $\eta = \frac{r}{\sqrt{\nu t}}$,

$$-\frac{f'\eta}{2} = f'' - \frac{f'}{\eta}, \quad (19)$$

with $r = R \rightarrow 0$ boundary condition yielding $f(\eta \rightarrow 0) = 1$. Both $r \rightarrow \infty$ and $t = 0$ conditions yield $f(\eta \rightarrow \infty) = 0$. Solving the above equations,

$$\ln f' + \frac{\eta^2}{4} - \ln \eta = \ln C_1 \implies f' = C_1 \eta e^{-\eta^2/4} \implies f = -2C_1 e^{-\eta^2/4} + C_2. \quad (20)$$

From boundary conditions, we obtain $C_2 = 0$ and $C_1 = -1/2$. $u_\theta(r, t)$ is obtained as,

$$u_\theta(r, t) = \frac{\Gamma}{2\pi r} e^{-r^2/4\nu t}. \quad (21)$$

Clearly, the above solution $u_\theta(r, t \rightarrow \infty) = \frac{\Gamma}{2\pi r} \lim_{t \rightarrow \infty} e^{-r^2/4\nu t} = \frac{\Gamma}{2\pi r}$, same as the steady state solution.