

PRACTICE SHEET-4

SOLUTIONS

$$\textcircled{1} \quad \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)^2 \psi = 0 \quad (E^2) E^2 \psi = 0$$

TO FIND GENERAL SOLUTIONS

$$\left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \psi = -\omega \quad \text{--- (1)} \quad E^2 \psi = -\omega$$

$$\left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \omega = 0 \quad \text{--- (2)}$$

SOLVE (2) FIRST LET $\omega = R(r) F(\theta)$

$$\frac{r^2 R'' + r R'}{R} + \frac{F''}{F} = 0 \quad \text{LET } \frac{F''}{F} = -\lambda^2 \quad \left(\begin{array}{l} \text{FULL PERIODIC} \\ \text{SOLUTIONS} \end{array} \right)$$

→ ANY (REAL / IMAGINARY) COMPLEX VALUE

$$\Rightarrow F = \sin(\lambda \theta), \cos(\lambda \theta)$$

$$\Rightarrow R = r^\lambda, r^{-\lambda} \quad \lambda \neq 0$$

FOR $\lambda = 0$ $F = C$, $R = K, \ln r$

$\downarrow$ CONSTANT $\downarrow$ CONSTANT

⇒ MOST GENERAL ω

$$= (a_0 + b_0 \theta) (c_0 + d_0 \ln r) + \sum_{n=1}^{\infty} (a_n \cos \lambda_n \theta + b_n \sin \lambda_n \theta) (c_n r^{\lambda_n} + d_n r^{-\lambda_n}) \quad \text{--- (3)}$$

TO SOLVE (1) (PARTICULAR SOLUTION)

TRY $\psi = r^s \sin \lambda_n \theta, r^s \cos \lambda_n \theta$

$$E^2 \psi = s(s-1) r^{s-2} \sin \lambda_n \theta + s r^{s-2} \sin \lambda_n \theta - \lambda_n^2 r^{s-2} \sin \lambda_n \theta$$

$$= (s^2 - \lambda_n^2) r^{s-2} \sin(\lambda_n \theta), (s^2 - \lambda_n^2) r^{s-2} \cos(\lambda_n \theta)$$

FROM (3) FOR $\lambda_n \neq 0, \neq 1$

$$s-2 = \lambda_n, -\lambda_n \Rightarrow s = \lambda_n + 2, -\lambda_n + 2$$

FOR $\lambda = 1$, $s = 3$ GIVES ONE SOLUTION SET

$$r^3 \sin \theta, r^3 \cos \theta$$

FOR $s = -\lambda + 2 = 1$ IS A HOMOGENEOUS SOLUTION
TO GET PARTICULAR SOLUTION, WE ASSUME
(FOR $\lambda = 1$)

$$r^s \theta \sin \theta, r^s \theta \cos \theta$$

$$\begin{aligned} \Rightarrow E^2 \psi &= s(s-1) r^{s-2} \theta \sin \theta + \cancel{s r^{s-2} \theta \sin \theta} \\ &\quad + r^{s-2} (\theta \cos \theta + r \sin \theta)' \\ &= s^2 r^{s-2} \theta \sin \theta + r^{s-2} (-\theta \sin \theta + 2 \cos \theta) \\ &= (s^2 - 1) r^{s-2} \theta \sin \theta + 2 r^{s-2} \cos \theta \end{aligned}$$

AND

$$\text{RHS} = (a \cos \theta + b \sin \theta) \left(c r + \frac{d}{r} \right)$$

$$\Rightarrow s = 1 \text{ FOR } \lambda = 1$$

AND CHARACTERISTIC SOLUTIONS ARE $r \theta \sin \theta, r \theta \cos \theta$

FOR $\lambda = 0$

$$E^2 \psi = (a + b \theta) (c + d \ln r)$$

$$\left(\frac{r^2 R'' + 4R'}{r^2} \right) f + \frac{R}{r^2} f'' = (a + b \theta) (c + d \ln r)$$

$$f'' = 0, \quad \frac{r^2 R'' + 4R'}{r^2} = c + d \ln r$$

$$\Rightarrow \frac{1}{r} \frac{d}{dr} \left(r \frac{dR}{dr} \right) = c + d \ln r$$

$$\Rightarrow \frac{dR}{dr} = \frac{c r}{2} + \frac{d}{2} \left(r \ln r - \frac{r}{2} \right) + K_1$$

$$\Rightarrow R = \hat{c} r^2 + \hat{d} (r^2 \ln r - r^2)$$

$\Rightarrow$ MOST GENERAL SOLUTION OF ψ IS

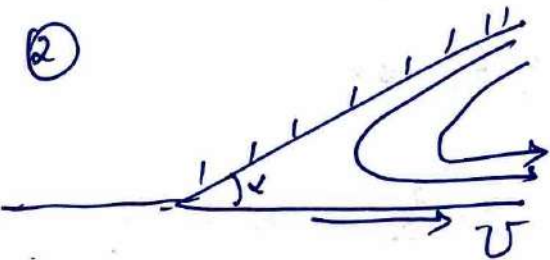
$$\psi = \cancel{c_0 + d}$$

$$\psi = \underbrace{\left[C_0 + d_0 \ln r + \hat{C} r^2 + \hat{d} r^2 (\ln r - 1) \right]}_{\text{HOMOGENEOUS}} \underbrace{\left[a_0 + b_0 \theta \right]}_{\text{HOMOGENEOUS/PARTICULAR}} + \underbrace{\left(C_1 r + d_1 r^{-1} + \hat{C}_1 r^3 \right)}_{\text{HOMOGENEOUS}} \underbrace{\left(a_1 r \sin \theta + b_1 \cos \theta \right)}_{\text{HOMOGENEOUS/PARTICULAR}} + \underbrace{\hat{d}_1 r}_{\text{PARTICULAR}} \underbrace{\left(\hat{a}_1 \theta \sin \theta + \hat{b}_1 \theta \cos \theta \right)}_{\text{PARTICULAR}} + \sum_{n=2}^{\infty} \underbrace{\left(C_n r^{\lambda_n} + d_n r^{-\lambda_n} \right)}_{\text{HOMOGENEOUS}} + \underbrace{\left(\hat{C}_n r^{\lambda_n+2} + \hat{d}_n r^{2-\lambda_n} \right)}_{\text{PARTICULAR}} \underbrace{\left(a_n \sin \lambda_n \theta + b_n \cos \lambda_n \theta \right)}_{\text{HOMOGENEOUS/PARTICULAR}}$$

FOR $r^m f_m(\theta)$ TYPE, WE HAVE.

$$r^{\pm 1} \sin \theta, r^{\pm 1} \cos \theta, r^3 \sin \theta, r^3 \cos \theta, r \theta \sin \theta, r \theta \cos \theta, r^{\pm \lambda_n} \sin \lambda_n \theta, r^{\pm \lambda_n} \cos \lambda_n \theta, r^{\pm \lambda_n+2} \sin \lambda_n \theta, r^{\pm \lambda_n+2} \cos \lambda_n \theta$$

②



a) $u(r, \theta=0) = U, u_r(r, \theta=\alpha) = 0$
NO SLIP

$u_\theta(r, \theta=0) = 0, u_\theta(r, \theta=\alpha) = 0$
HARD WALL.

$$u_r = -\frac{1}{r} \frac{\partial \psi}{\partial \theta}, u_\theta = \frac{\partial \psi}{\partial r}$$

$$\Rightarrow \left. \frac{1}{r} \frac{\partial \psi}{\partial \theta} \right|_{\theta=0, \alpha} = (U, 0), \left. \frac{\partial \psi}{\partial r} \right|_{\theta=0, \alpha} = 0$$

b) For $\psi = r^m f_m(\theta)$ $r^{m-1} f'(\theta=0, \alpha) = 0$ FOR ALL r

$$\Rightarrow m=1$$

$$\Rightarrow \psi = r f(\theta)$$

SUBSTITUTING IN $\nabla^2(\nabla^2 \psi) = 0$ (OR JUST REFER TO FINAL ANSWER FOR SOLUTIONS PROPORTIONAL TO r)

$$\psi = A(r \sin \theta + B \cos \theta + C \theta \sin \theta + D \theta \cos \theta)$$

$$\frac{1}{r} \frac{\partial \psi}{\partial \theta} \Big|_{\theta=0} = U \Rightarrow A + D = U$$

$$\frac{1}{r} \frac{\partial \psi}{\partial \theta} \Big|_{\theta=\alpha} = 0 \Rightarrow A \cos \alpha - B \sin \alpha + C(\sin \alpha + \alpha \cos \alpha) + D(\cos \alpha - \alpha \sin \alpha) = 0$$

$$\frac{\partial \psi}{\partial r} \Big|_{r=0} = 0 \Rightarrow B = 0$$

$$\frac{\partial \psi}{\partial r} \Big|_{r=\alpha} = 0 \Rightarrow A \sin \alpha + B \cos \alpha + C \alpha \sin \alpha + D \alpha \cos \alpha = 0$$

$$B = 0, D = U - A$$

$$A \cos \alpha + C(\sin \alpha + \alpha \cos \alpha) + (U - A)(\cos \alpha - \alpha \sin \alpha) = 0$$

$$A \sin \alpha + C \alpha \sin \alpha + (U - A) \alpha \cos \alpha = 0$$

$$\text{OR} \quad A \alpha \sin \alpha + U \cos \alpha - U \alpha \sin \alpha + C(\sin \alpha + \alpha \cos \alpha) = 0$$

$$A \sin \alpha + C \alpha \sin \alpha + (U - A) \alpha \cos \alpha = 0$$

SOLVING

$$C = \frac{U(\alpha - \sin \alpha \cos \alpha)}{\sin^2 \alpha - \alpha^2}, \quad D = \frac{U \sin^2 \alpha}{\sin^2 \alpha - \alpha^2}, \quad A = \frac{U \alpha^2}{\sin^2 \alpha - \alpha^2}$$

$$\psi = -\frac{U \alpha}{(\sin^2 \alpha - \alpha^2)} \left[\alpha^2 \sin \theta + (\sin \alpha \cos \alpha - \alpha) \theta \sin \theta - (\sin^2 \alpha) \theta \cos \theta \right]$$

$$c) \quad \tau_{\theta\theta} = \mu \left(r \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right) + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) \Big|_{\theta=0}$$

$$u_\theta = \frac{\partial \psi}{\partial r}, \quad u_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$$

$$\tau_{\theta\theta} \Big|_{\theta=0} = -\frac{2\mu U}{r(\sin^2 \alpha - \alpha^2)} (\sin \alpha \cos \alpha - \alpha)$$

$$(3) \quad \nabla^2 \underline{u} - \nabla p = \underline{f} \delta(\underline{x} - \underline{y}) \quad (1), \quad \nabla \cdot \underline{u} = 0$$

TAKE DIVERGENCE

$$\nabla \cdot (\nabla^2 \underline{u}) - \nabla^2 p = \nabla \cdot (\underline{f} \delta(\underline{x} - \underline{y})) \quad (2)$$

$$p(\underline{x}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \hat{p}(\underline{k}) e^{i\underline{k} \cdot \underline{x}} d\underline{k} \quad \nabla^2 p = -\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} k^2 \hat{p}(\underline{k}) e^{i\underline{k} \cdot \underline{x}} d\underline{k}$$

$$\nabla \cdot (\underline{f} \delta(\underline{x} - \underline{y})) = \frac{1}{(2\pi)^3} \nabla_{\underline{x}} \cdot \int \underline{f} e^{i\underline{k} \cdot (\underline{x} - \underline{y})} d\underline{k} = \frac{i}{(2\pi)^3} \int \underline{k} \cdot \underline{f} e^{i\underline{k} \cdot (\underline{x} - \underline{y})} d\underline{k}$$

SUBSTITUTING IN (2)

$$\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} k^2 \hat{p}(\underline{k}) e^{i\underline{k} \cdot \underline{x}} d\underline{k} = \frac{1}{(2\pi)^3} \int i\underline{k} \cdot \underline{f} e^{i\underline{k} \cdot \underline{x}} e^{-i\underline{k} \cdot \underline{y}} d\underline{k}$$

$$\Rightarrow k^2 \hat{p}(\underline{k}) = i\underline{k} \cdot \underline{f} e^{-i\underline{k} \cdot \underline{y}} \quad \text{OR} \quad \hat{p}(\underline{k}) = \frac{i\underline{k} \cdot \underline{f} e^{-i\underline{k} \cdot \underline{y}}}{k^2}$$

NOW CONSIDER (1)

$$\begin{aligned} \nabla p &= \frac{\nabla}{(2\pi)^3} \int_{\mathbb{R}^3} \hat{p}(\underline{k}) e^{i\underline{k} \cdot \underline{x}} d\underline{k} = \frac{i}{(2\pi)^3} \int_{\mathbb{R}^3} \underline{k} \hat{p}(\underline{k}) e^{i\underline{k} \cdot \underline{x}} d\underline{k} \\ &= -\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\underline{k} (\underline{k} \cdot \underline{f})}{k^2} e^{i\underline{k} \cdot (\underline{x} - \underline{y})} d\underline{k} \end{aligned}$$

$$\nabla^2 \underline{u} = -\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} k^2 \hat{u}(\underline{k}) e^{i\underline{k} \cdot \underline{x}} d\underline{k} \quad \text{SUBSTITUTING}$$

$$\begin{aligned} -\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} k^2 \hat{u}(\underline{k}) e^{i\underline{k} \cdot \underline{x}} d\underline{k} + \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\underline{k} (\underline{k} \cdot \underline{f})}{k^2} e^{i\underline{k} \cdot (\underline{x} - \underline{y})} d\underline{k} \\ = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \underline{f} e^{i\underline{k} \cdot (\underline{x} - \underline{y})} d\underline{k} \end{aligned}$$

$$k^2 \hat{u}(\underline{k}) = \frac{\underline{k}(\underline{k} \cdot \underline{f})}{k^2} e^{-i\underline{k} \cdot \underline{y}} - \underline{f} e^{-i\underline{k} \cdot \underline{y}}$$

$$\hat{u}(\underline{k}) = \frac{\underline{k}(\underline{k} \cdot \underline{f})}{k^4} e^{-i\underline{k} \cdot \underline{y}} - \frac{\underline{f}}{k^2} e^{-i\underline{k} \cdot \underline{y}} \quad \textcircled{I} \quad \textcircled{II}$$

$$\underline{u}(\underline{x}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \hat{u}(\underline{k}) e^{i\underline{k} \cdot \underline{x}} d\underline{k}$$

$$\textcircled{II} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \frac{\underline{f}}{k^2} e^{i\underline{k} \cdot (\underline{x} - \underline{y})} d\underline{k} = \frac{\underline{f}}{4\pi |\underline{x} - \underline{y}|}$$

$$\textcircled{I} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\underline{k}(\underline{k} \cdot \underline{f})}{k^4} e^{i\underline{k} \cdot (\underline{x} - \underline{y})} d\underline{k} \quad \text{IN INDEX NOTATION}$$

$$= \frac{f_j}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{k_i k_j}{k^4} e^{i\underline{k} \cdot (\underline{x} - \underline{y})} d\underline{k}$$

IDENTIFY

$$= \int_{\mathbb{R}^3} \frac{k_i k_j}{k^4} e^{i\underline{k} \cdot (\underline{x} - \underline{y})} d\underline{k} = \frac{\partial^2}{\partial x_i \partial x_j} \int_{\mathbb{R}^3} \frac{e^{i\underline{k} \cdot (\underline{x} - \underline{y})}}{k^4} d\underline{k}$$

$$\Rightarrow \textcircled{I} = -\frac{f_j}{(2\pi)^3} \frac{\partial^2}{\partial x_i \partial x_j} \left(\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{i\underline{k} \cdot (\underline{x} - \underline{y})}}{k^4} d\underline{k} \right)$$

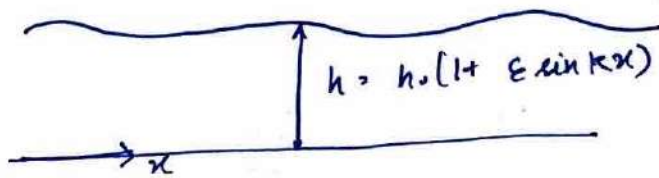
$$= + f_j \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{8\pi}.$$

$$\frac{\partial^2 |\underline{x} - \underline{y}|}{\partial x_i \partial x_j} = \frac{\delta_{ij}}{r} - \frac{(x_i - y_i)(x_j - y_j)}{r^3} \quad (\text{VERIFY})$$

$$\Rightarrow \textcircled{I} = \frac{f_j}{8\pi r} - \frac{(x_i - y_i)(x_j - y_j) f_j}{8\pi r^3}$$

$$\Rightarrow \underline{u}(\underline{x}) = - \left(\frac{\underline{f}}{8\pi r} + \frac{\underline{r}(\underline{f} \cdot \underline{r})}{8\pi r^3} \right) \quad \text{WHERE } \underline{r} = \underline{x} - \underline{y}$$

④



For $\frac{h_0}{L} \ll 1 \Rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} \frac{dp}{dx}$

LENGTH SCALE FOR x
 $[x] = \frac{1}{k}$

$\Rightarrow k h_0 \ll 1 \Rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} \frac{dp}{dx}$

$u = \frac{y^2}{2\mu} \frac{dp}{dx} + C_1 y + C_2$ $u(y, 0) = 0 \Rightarrow C_2 = 0$

$C_1 h + \frac{h^2}{2\mu} \frac{dp}{dx} = 0 \Rightarrow C_1 = -\frac{h}{2\mu} \frac{dp}{dx}$

$u(x, y) = -\frac{h^2}{2\mu} \frac{dp}{dx} \left(\frac{y}{h} - \frac{y^2}{h^2} \right)$

$Q = \int_0^h u dy = -\frac{1}{2\mu} \frac{dp}{dx} \int_0^h (hy - y^2) dy = -\frac{1}{2\mu} \frac{dp}{dx} \left[\frac{h^3}{2} - \frac{h^3}{3} \right]$

$Q = -\frac{h^3}{12\mu} \frac{dp}{dx} \Rightarrow -\frac{dp}{dx} = \frac{12\mu Q}{h^3}$

$h = h_0(1 + \epsilon \sin(kx))$
 $h^3 = h_0^3(1 + \epsilon \sin(kx))^3$

For $\epsilon \ll 1$ $h^3 \approx h_0^3(1 + 3\epsilon \sin(kx))$

$\Rightarrow -\frac{dp}{dx} \approx \frac{12\mu Q}{h_0^3} \frac{1}{1 + 3\epsilon \sin(kx)}$

$$a) \quad \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \underline{u}, \quad \nabla \cdot \underline{u} = 0$$

$$[\underline{u}] = U_0, \quad [x, y, z] = a, \quad [t] = \frac{1}{\omega}$$

$$[p] = p_0$$

$$U_0 \omega \frac{\partial \underline{u}^*}{\partial t^*} + \frac{U_0^2}{a} \underline{u}^* \cdot \nabla^* \underline{u}^* = -\frac{p_0}{\rho a} \nabla^* p^* + \frac{\nu U_0}{a^2} \nabla^{*2} \underline{u}^*$$

$$\& \quad \nabla^* \cdot \underline{u}^* = 0$$

DROPPING THE STARS & CHOOSING $p_0 = \frac{\rho \nu U_0}{a}$

$$\frac{a^2 \omega}{\nu} \frac{\partial \underline{u}}{\partial t} + \frac{U_0 a}{\nu} (\underline{u} \cdot \nabla \underline{u}) = -\nabla p + \nabla^2 \underline{u}$$

$$b) \quad \text{For } \frac{U_0 a}{\nu} \ll 1 \quad \& \quad \frac{a^2 \omega}{\nu} \sim 1$$

$$\frac{\partial \underline{u}}{\partial t} + \nabla p = \nabla^2 \underline{u} \quad \rightarrow \text{UNSTEADY STOKES FLOW (DIMENSIONLESS)}$$

$$\frac{\partial \underline{u}}{\partial t} + \frac{\nabla p}{\rho} = \nu \nabla^2 \underline{u} \quad (\text{DIMENSIONAL})$$

$$c(i) \quad i \frac{\partial \underline{u}}{\partial t} + \frac{\nabla p}{\rho} = \nu \nabla^2 \underline{u}$$

$$(ii) \quad \nabla \times \left(i \frac{\partial \underline{u}}{\partial t} + \frac{\nabla p}{\rho} \right) = \nu \nabla \times (\nabla^2 \underline{u})$$

$$\text{SINCE } \frac{\partial \underline{u}}{\partial \phi} = 0 \quad \& \quad u_\phi = 0 \quad \underline{\omega} = \nabla \times \underline{u} \quad (\text{GIVEN RELATION})$$

$$= \frac{1}{r} \left(\frac{\partial}{\partial r} (ru_\theta) - \frac{\partial u_r}{\partial \theta} \right) \hat{e}_\phi$$

$$i\Omega \hat{\omega}_\phi = \gamma \left[\nabla^2 (\nabla \times \hat{u}) \right] \phi$$

$$= \gamma \left(\nabla^2 \hat{\omega}_\phi - \frac{\hat{\omega}_\phi}{r^2 \sin^2 \theta} \right)$$

$$= \gamma \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \hat{\omega}_\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial (\hat{\omega}_\phi \sin \theta)}{\partial \theta} \right) \right]$$

(GIVEN RELATION)

$$(iii) \quad \hat{u}_r = f(r) \cos \theta, \quad \hat{u}_\theta = g(r) \sin \theta$$

$$\nabla \cdot \hat{u} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \hat{u}_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\hat{u}_\theta \sin \theta) = 0$$

$$\frac{1}{r^2} \frac{d}{dr} (r^2 f) \cos \theta + \frac{2g}{r} \cos \theta = 0$$

$$\Rightarrow \boxed{g(r) = -\frac{1}{2r} \frac{d}{dr} (r^2 f)}$$

$$(iv) \quad \omega_\phi = \frac{1}{r} \left(\frac{\partial}{\partial r} (r u_\theta) - \frac{\partial u_r}{\partial \theta} \right)$$

$$= \frac{1}{r} \left(\frac{d}{dr} (r g(r)) + f \right) \sin \theta$$

$$= \frac{1}{r} \left[-\frac{1}{2} \frac{d}{dr} (2rf + r^2 f') + f \right] \sin \theta = \frac{1}{r} \left[-\frac{1}{2} (2rf' + 2f + 2rf' + r^2 f'') + f \right]$$

$$= -\frac{\sin \theta}{2r} \left[\underbrace{r^2 f'' + 4rf'}_{F(r)} \right] = -\frac{\sin \theta}{2} \underbrace{\left[rf'' + 4f' \right]}_{F(r)}$$

$$(v) \quad -\sin \theta \frac{i\Omega}{\gamma} F(r) = \gamma \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{dF}{dr} \right) (-\sin \theta) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial (-\sin^2 \theta)}{\partial \theta} \right) F(r) \right]$$

$$-\sin \theta \frac{i\Omega}{\gamma} F(r) = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dF}{dr} \right) (-\sin \theta) + \frac{F(r)}{r^2} \frac{\partial}{\partial \theta} \left(\frac{-2\cos \theta \sin \theta}{\sin \theta} \right)$$

$$-\sin\theta \frac{i\Omega}{r} F(\eta) = -\sin\theta \frac{d}{d\eta} \left(\eta^2 F'(\eta) \right) + 2\sin\theta \frac{F(\eta)}{\eta^2}$$

$$\Rightarrow \frac{i\Omega}{r} F = \frac{1}{\eta^2} \frac{d}{d\eta} (\eta^2 F') - \frac{2F}{\eta^2}$$

$$\frac{i\Omega}{r} F = F'' + \frac{2F'}{\eta} - \frac{2F}{\eta^2}$$

$$\eta^2 F'' + 2\eta F' - 2F = \frac{i\Omega}{r} \eta^2 F$$

$$F = \frac{1}{2} (\eta f'' + 4f') \quad \text{FOR } f = -\frac{2h'}{\eta}, \quad f' = \frac{2h''}{\eta^2} - \frac{2h'''}{\eta}$$

$$f'' = -\frac{2h'}{\eta^3} + \frac{2h''}{\eta^2} + \frac{2h''}{\eta^2} - \frac{2h'''}{\eta}$$

$$= -\frac{2h'''}{\eta} - \frac{4h'}{\eta^3} + \frac{4h''}{\eta^2} \Rightarrow \eta f'' + 4f'$$

$$= -\frac{2h'''}{\eta} + \frac{4h''}{\eta^2} - \frac{4h'}{\eta^3} + \frac{8h''}{\eta^2} - \frac{8h'''}{\eta}$$

$$= -2h''' + \frac{4h''}{\eta} - \frac{4h'}{\eta^3} + \frac{8h''}{\eta^2} - \frac{8h'''}{\eta}$$

$$= -2 \left(h''' + \frac{2h''}{\eta} - \frac{2h'}{\eta^2} \right)$$

GIVEN RELATION

$$\frac{1}{\eta} \frac{d}{d\eta} (\nabla_\eta^2 f) = f''' - \frac{2f'}{\eta^2} + \frac{2f''}{\eta}$$

$$\Rightarrow \frac{i\Omega}{r} \frac{d}{d\eta} (\nabla_\eta^2 h) = -\frac{d}{d\eta} (\nabla_\eta^2 (\nabla_\eta^2 h))$$

$$\Rightarrow \frac{i\Omega}{r} \nabla_\eta^2 h + \nabla_\eta^2 (\nabla_\eta^2 h) = \text{const.}$$

$\eta \rightarrow \infty$, FLOW IS QUIESCENT $\Rightarrow$ ALL DERIVATIVES OF $u \rightarrow 0$

SOLVE FOR $h(r)$

$$\nabla_r^2 h = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dh}{dr} \right) = \frac{1}{r^2} \left(r^2 h'' + 2r h' \right)$$

$$= h'' + \frac{2h'}{r}$$

$$\Rightarrow \frac{i\Omega}{\gamma} \left(h'' + \frac{2h'}{r} \right) + (\nabla_r^2 h)'' + 2 \left(\frac{\nabla_r^2 h}{r} \right)' = 0$$

$$\text{LET } \nabla_r^2 h = G(r)$$

$$i\Omega G + G'' + \frac{2G'}{r} = 0 \Rightarrow r^2 G'' + 2r G' + \frac{i\Omega}{\gamma} r^2 G = 0$$

$$\text{LET } \frac{i\Omega}{\gamma} = k^2 \quad \& \quad G = \frac{m(r)}{r} \Rightarrow G' = \frac{m'}{r} - \frac{m}{r^2}$$

$$\& \quad G'' = \frac{m''}{r} - \frac{2m'}{r^2} + \frac{2m}{r^3}$$

$$r^2 G'' + 2r G' + k^2 G = 0 \Rightarrow \frac{m''}{r} - \frac{2m'}{r^2} + \frac{2m}{r^3} + \frac{2}{r} \left(\frac{m'}{r} - \frac{m}{r^2} \right) + k^2 \frac{m}{r} = 0$$

$$\Rightarrow m'' + k^2 m = 0 \Rightarrow m(r) = e^{\pm ikr}$$

$$\Rightarrow G(r) = \frac{AC e^{\pm ikr}}{r}$$

$$\nabla_r^2 h = G(r) \Rightarrow h'' + \frac{2h'}{r} = \frac{AC e^{\pm ikr}}{r}$$

$$\text{OR } \frac{d}{dr} (r^2 h') = A r e^{\pm ikr} \Rightarrow r^2 h' = A \left(\frac{r^2}{2} e^{\pm ikr} + \frac{1}{k^2} e^{\pm ikr} \right)$$

$$\Rightarrow h' = \frac{A}{2} e^{\pm ikr} + \frac{A}{k^2 r^2} e^{\pm ikr}$$

NO NEED TO SOLVE FOR h'

$$f = \frac{-2h'}{h} = -\frac{2A}{h} \left(\frac{e^{\pm ika}}{2} + \frac{1}{k^2 h^2} e^{\pm ika} \right)$$

SIMILARLY GET $g(h)$ FROM

$$g(h) = -\frac{1}{2h} \frac{d}{dh} (h^2 f)$$