

PRACTICE SHEET 5
SOLUTIONS

① $\varepsilon \frac{d^2 u}{dy^2} + \frac{du}{dy} = 0 \quad u(0) = 0, \quad u(1) = 1$

$\varepsilon \ll 0$ OUTER SOLUTION $\frac{du}{dy} = 0 \rightarrow u = \text{const.}$

LET $\tilde{y} = \frac{y}{\varepsilon} \quad \frac{du}{dy} = \frac{1}{\varepsilon} \frac{du}{d\tilde{y}}, \quad \frac{d^2 u}{dy^2} = \frac{1}{\varepsilon^2} \frac{d^2 u}{d\tilde{y}^2}$

$\Rightarrow \frac{d^2 u}{d\tilde{y}^2} + \frac{du}{d\tilde{y}} = 0 \quad \Rightarrow \frac{d^2 u}{d\tilde{y}^2} + u = c$

$\Rightarrow -\ln(c-u) = \tilde{y} - \ln k$

$u = C - K e^{-\tilde{y}}$

$\frac{K}{C-u} = e^{\tilde{y}}$

$= C - K e^{-y/\varepsilon}$

FOR SMALL y , $\tilde{y} \sim O(1)$

$u = \text{const} = 1 \Rightarrow u(1) = 1$

$u(0) = 0 \Rightarrow C - K = 0 \Rightarrow u = C(1 - e^{-y/\varepsilon})$ FOR SMALL y .

$u = C(1 - e^{-y/\varepsilon})$ FOR $y \rightarrow \infty$ (OR $\varepsilon \rightarrow 0$) $u = C$

$\Rightarrow C = 1 \Rightarrow u = 1 - e^{-y/\varepsilon}$

EXACT SOLUTION

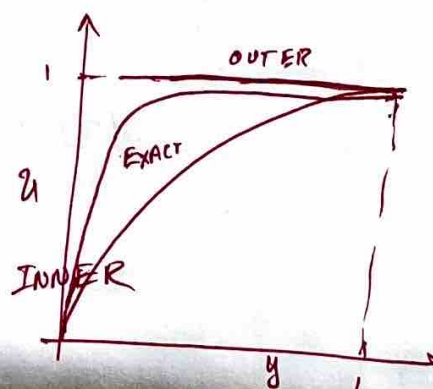
$\varepsilon \frac{d^2 u}{dy^2} + u = C_1 \Rightarrow \frac{du}{C_1 - u} = \frac{dy}{\varepsilon} \Rightarrow \ln \frac{C_1}{C_1 - u} = \frac{y}{\varepsilon}$

$\Rightarrow u = C_1 - C_2 e^{-y/\varepsilon}$

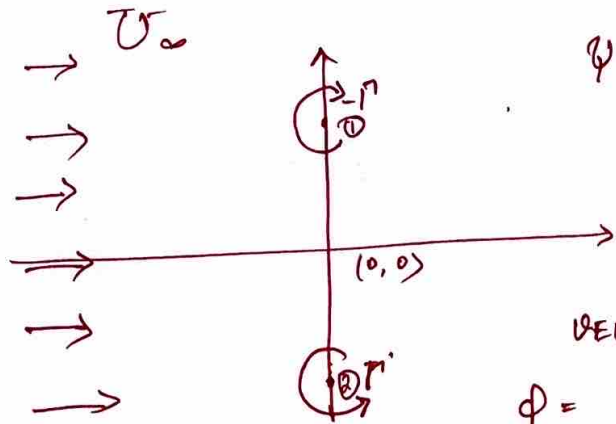
$u(y=0) = 0 \Rightarrow C_1 = C_2$

$u(y=1) = 1 \Rightarrow C_1 - C_1 e^{-1/\varepsilon} = 1 \Rightarrow C_1 = \frac{1}{1 - e^{-1/\varepsilon}}$

$\Rightarrow u = \frac{1 - e^{-y/\varepsilon}}{1 - e^{-1/\varepsilon}}$



①



$$\psi = \psi_{\text{UNIFORM}} + \psi_1 + \psi_2$$

$$= U_{\infty} y + \frac{\Gamma}{2\pi} \ln \sqrt{x^2 + (y-a)^2} - \frac{\Gamma}{2\pi} \ln \sqrt{x^2 + (y+a)^2}$$

VELOCITY POTENTIAL

$$\phi = U_{\infty} x + \frac{\Gamma}{2\pi} \tan^{-1} \frac{y-a}{x} - \frac{\Gamma}{2\pi} \tan^{-1} \frac{y+a}{x}$$

STAGNATION POINTS

→ WHERE $u, v = 0$. ~~SIMPLE IN ϕ~~

$$u = \frac{\partial \psi}{\partial y} = U_{\infty} + \frac{\Gamma}{2\pi} \left(\frac{1}{\sqrt{x^2 + (y-a)^2}} \right) \left(\frac{1}{2\sqrt{x^2 + (y-a)^2}} \right) (2y-a) - \frac{\Gamma}{2\pi} \left(\frac{2(y+a)}{2(x^2 + (y+a)^2)} \right)$$

$$= U_{\infty} + \frac{\Gamma}{2\pi} \left[\frac{y-a}{x^2 + (y-a)^2} - \frac{y+a}{x^2 + (y+a)^2} \right]$$

$$v = \frac{\partial \psi}{\partial x} = \frac{\Gamma}{2\pi} \left[\frac{x}{x^2 + (y-a)^2} - \frac{x}{x^2 + (y+a)^2} \right]$$

$$v = 0 \Rightarrow x(x^2 + (y+a)^2) = x(x^2 + (y-a)^2) \Rightarrow x = 0 \text{ AND } y = 0 \text{ ARE SOLUTIONS}$$

FOR $y = 0$ $u = U_{\infty} + \frac{\Gamma}{2\pi} \left[\frac{-a}{x^2 + a^2} - \frac{a}{x^2 + a^2} \right] = 0$

$$u = 0 \Rightarrow U_{\infty} = \frac{\Gamma a}{\pi(x^2 + a^2)} \Rightarrow x = \pm a \sqrt{\frac{\Gamma}{\pi U_{\infty}} - 1}, y = 0$$

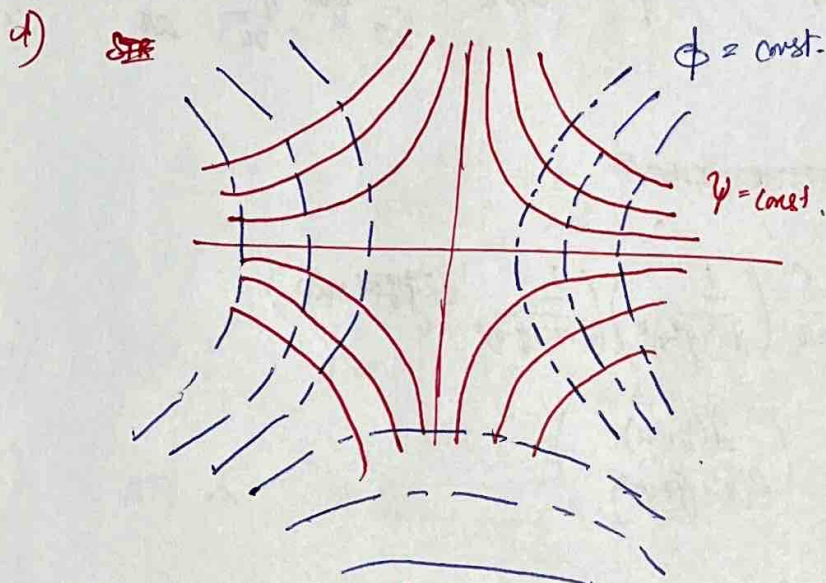
~~$x = 0$~~ SIMILARLY FOR $x = 0$, $u = 0 \Rightarrow y = \pm a \sqrt{1 - \frac{\Gamma}{\pi U_{\infty} a}}$

② $\phi = x^2 - y^2$, $\underline{u} = \nabla \phi$ a) $u = \frac{\partial \phi}{\partial x} = 2x$, $v = \frac{\partial \phi}{\partial y} = -2y$

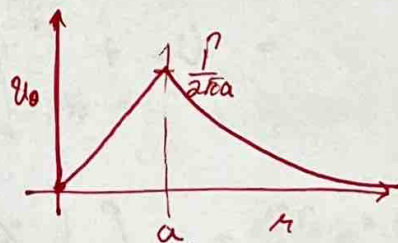
b) $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2 - 2 = 0$, $\nabla \times \underline{u} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$

c) $\frac{\partial \psi}{\partial y} = u$, $v = -\frac{\partial \psi}{\partial x} \Rightarrow \frac{\partial \psi}{\partial y} = 2x \Rightarrow \psi = 2xy + c(x)$

$v = -\frac{\partial \psi}{\partial x} = -2y + \frac{dc}{dx} = -2y \Rightarrow \psi = 2xy$



③ $u_0 = \frac{\Gamma}{2\pi} \begin{cases} r/a^2 & r \leq a \\ 1/r & r > a \end{cases}$



$\underline{\omega} = \omega \hat{k} = \frac{1}{r} \frac{\partial}{\partial r} (r u_\theta) - \frac{1}{r} \frac{\partial u_r}{\partial \theta}$

$\nabla \cdot \underline{u} = 0 \Rightarrow \frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} = 0 \Rightarrow r u_r = \text{const} \quad (u_\theta = 0 \text{ GIVEN})$

$\omega = \frac{1}{r} \frac{\partial}{\partial r} (r u_\theta) = \begin{cases} \frac{\Gamma}{\pi a^2} & r \leq a \\ 0 & r > a \end{cases}$

$u_\theta = -\frac{\partial \psi}{\partial r}$

$\psi = -\frac{\Gamma}{2\pi} \begin{cases} \frac{r^2}{2a^2} & r \leq a \\ \ln r & r > a \end{cases}$

$\frac{1}{r} \frac{\partial \phi}{\partial \theta} = u_\theta \Rightarrow \phi = \frac{\Gamma}{2\pi} \begin{cases} \frac{r^2 \theta}{a^2} & r \leq a \\ \theta & r > a \end{cases}$



SINK + IDEAL VORTEX

$$\Rightarrow \phi = -\frac{q}{2\pi} \ln r - \frac{k}{2\pi} \theta \rightarrow \text{C.W. VORTEX}$$

$$p(r_0) - p_\infty = -p_0 \quad (p_0 = 2200 \text{ Pa})$$

a) $u_r = \frac{1}{r} \frac{\partial \phi}{\partial \ln r} = -\frac{q}{2\pi r}, \quad u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{k}{2\pi r}$

$$p(r_0) + \frac{1}{2} \rho \left(\left(\frac{q}{2\pi r} \right)^2 + \left(\frac{k}{2\pi r} \right)^2 \right) = p_\infty$$

$$\frac{2p_0}{\rho} = \frac{1}{4\pi^2} \left(\frac{q^2}{r_0^2} + \frac{k^2}{r_0^2} \right) \Rightarrow \sqrt{\frac{8\pi^2 p_0}{\rho} - \frac{k^2}{r_0^2}} = q$$

USE $\rho = 1 \text{ kg/m}^3$

b) $p(r) = p_\infty - \frac{1}{2} \rho \left(\left(\frac{q}{2\pi r} \right)^2 + \left(\frac{k}{2\pi r} \right)^2 \right)$

c) AT r_0 $\tan \theta = \frac{u_r}{u_\theta} = \frac{q}{k} \quad \theta = \tan^{-1} \left(\frac{q}{k} \right)$

⑤ $\phi = U_\infty r \cos \theta + \frac{U_\infty a^2 \cos \theta}{r} - \frac{\Gamma \theta}{2\pi}, \quad u_r = \frac{\partial \phi}{\partial r} = U_\infty \cos \theta - \frac{U_\infty a^2 \cos \theta}{r^2}$

$$u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -U_\infty \sin \theta + \frac{U_\infty a^2 \sin \theta}{r^2} - \frac{\Gamma}{2\pi r}$$

PROBLEM IN QUESTION $\phi_{\text{DOUBLET}} = \frac{U_\infty a^2 \cos \theta}{r}$

a) AT CYLINDER SURFACE $u_r = 0 \Rightarrow r = a \rightarrow \text{EIL}$

$$u_\theta(r=a) = -U_\infty \sin \theta + \frac{U_\infty a^2 \sin \theta}{a^2} - \frac{\Gamma}{2\pi a} = a\omega$$

$$\Rightarrow \omega = \frac{\Gamma}{2\pi a^2} - 2U_\infty \sin \theta - \frac{\Gamma}{2\pi a} = a\omega \xrightarrow{U_\infty}$$



b) $u_\theta = 0$ FOR $\sin \theta = -\frac{\Gamma}{4\pi U_\infty a}$

STAGNATION POINTS

$$r=a, \quad \theta = \sin^{-1} \left(-\frac{\Gamma}{4\pi U_\infty a} \right)$$

$$c) \quad p_{\infty} + \frac{1}{2} \rho U_{\infty}^2 = p_s + \frac{1}{2} \rho (U_n^2(r=a) + U_{\theta}^2(r=a))$$

$$p_{\infty} + \frac{1}{2} \rho U_{\infty}^2 = p_s + \frac{1}{2} \rho \left(\frac{\Gamma^2}{4\pi^2 a^2} + 2U_{\infty} \sin \theta \right)^2$$

$$p_s = p_{\infty} + \frac{1}{2} \rho U_{\infty}^2 \left(1 - 4 \sin^2 \theta - \frac{2\Gamma}{\pi U_{\infty} a} \sin \theta - \frac{\Gamma^2}{4\pi^2 U_{\infty}^2 a^2} \right)$$

$$d) \quad \text{DRAG} \quad F_D = - \int p \, dA \cos \theta \quad \left(\text{pressure force} = - \int p \hat{n} \, dA \right)$$

$$= -L \int_0^{2\pi} p a \, d\theta \cos \theta \quad L \rightarrow \text{LENGTH OF CYLINDER}$$

INTEGRATION OF $\int_0^{2\pi} \cos \theta \, d\theta = 0$, $\int_0^{2\pi} \sin^2 \theta \cos \theta \, d\theta = 0$

$$\int_0^{2\pi} \sin \theta \cos \theta \, d\theta = \frac{1}{2} \int_0^{2\pi} \sin 2\theta \, d\theta = 0$$

$$\Rightarrow F_D = 0$$

LIFT $F_L = -L \int_0^{2\pi} p a \, d\theta \sin \theta$

$$= \frac{L}{\pi} \rho U_{\infty}^2 \frac{\Gamma}{U_{\infty} a} \int_0^{2\pi} \sin^2 \theta \, d\theta$$

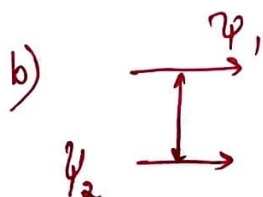
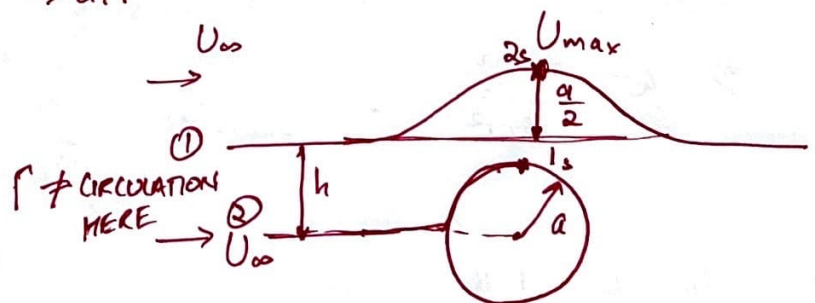
$$= (\rho U_{\infty} \Gamma) \frac{L}{\pi} \rightarrow \text{LIFT}$$

$$\int_0^{2\pi} \sin \theta \, d\theta = 0$$

$$\int_0^{2\pi} \sin^3 \theta \, d\theta = 0$$

$$6) \quad a) \quad \psi = U_{\infty} r \sin \theta - \frac{\Gamma}{2\pi h} \sin \theta$$

$$\phi = U_{\infty} r \cos \theta + \frac{\Gamma}{2\pi h} \cos \theta$$



$$\psi_1 - \psi_2 = \text{flow rate} \rightarrow \psi_1 - \psi_2 = U_{\infty} h \quad \text{LET } \psi_{\infty} = 0$$

$$\psi(r=a) = U_{\infty} a \sin \theta - \frac{\Gamma}{2\pi h} \sin \theta = 0$$

$$\Rightarrow a = \sqrt{\frac{\Gamma}{2\pi U_\infty}} \quad \text{or} \quad \Gamma = 2\pi a^2 U_\infty$$

$$\Rightarrow \psi = U_\infty r \sin\theta - \frac{U_\infty a^2 \sin\theta}{r}$$

$$\psi \left(r = h + \frac{a}{2} \right) = U_\infty \left(h + \frac{a}{2} \right) \sin\theta - \frac{U_\infty a^2 \sin\theta}{h + \frac{a}{2}}$$

$$\text{for } \theta = \pi/2 \quad \psi_{2S} = U_\infty \left(h + \frac{a}{2} \right) - \frac{U_\infty a^2}{h + \frac{a}{2}}$$

$$\Rightarrow \cancel{U_\infty h} = U_\infty \left(h + \frac{a}{2} \right) - \frac{U_\infty a^2}{h + \frac{a}{2}} \Rightarrow \frac{\cancel{U_\infty a}}{2} = \frac{U_\infty a^2}{h + \frac{a}{2}}$$

$$\boxed{h = \frac{3a}{2}}$$

$$c) \quad U_{\max} = \left. \frac{1}{r} \frac{\partial \phi}{\partial \theta} \right|_{\theta=\pi/2} = - \frac{\Gamma \sin\theta}{2\pi r^2} - \frac{U_\infty \sin\theta}{r} = - \frac{U_\infty a^2}{r^2} \Big|_{r=2a} \Rightarrow U_\infty$$

$$= - \cancel{4U_\infty} \left[- \frac{5U_\infty}{4} \right]$$

$$d) \quad C_p = \frac{p - p_\infty}{\frac{1}{2} \rho U_\infty^2}$$

$$p + \frac{1}{2} \rho (u_r^2 + u_\theta^2) = p_\infty + \frac{1}{2} \rho U_\infty^2$$

$$C_p = \frac{p - p_\infty}{\frac{1}{2} \rho U_\infty^2} = 1 - \frac{u_r^2 + u_\theta^2}{U_\infty^2}$$

$$u_r = \frac{\partial \phi}{\partial r} = U_\infty \cos\theta - \frac{\Gamma}{2\pi r^2} \cos\theta$$

$$\psi = U_\infty r \sin\theta - \frac{U_\infty a^2 \sin\theta}{r}$$

$$= \frac{3U_\infty a}{2}$$

$$u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = U_\infty \cos\theta \left(1 - \frac{a^2}{r^2} \right) = -U_\infty \sin\theta - \frac{U_\infty a^2 \sin\theta}{r^2} = -U_\infty \sin\theta \left(1 + \frac{a^2}{r^2} \right)$$

$$\Rightarrow \boxed{r \sin\theta - \frac{a^2 \sin\theta}{r} = \frac{3a}{2}} \rightarrow \text{SHAPE OF BUMP}$$

$$C_p = 1 - \left[\cos^2\theta \left(1 - \frac{a^2}{r^2} \right)^2 + \sin^2\theta \left(1 + \frac{a^2}{r^2} \right)^2 \right]$$

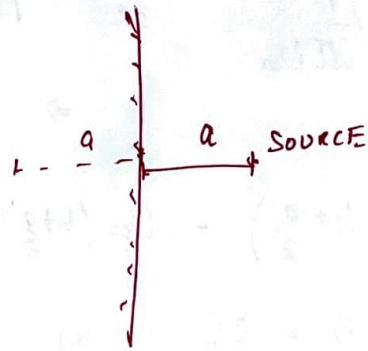
$$= 1 - \left[\cos^2\theta \left(1 + \frac{a^4}{r^4} - \frac{2a^2}{r^2} \right) + \sin^2\theta \left(1 + \frac{a^4}{r^4} + \frac{2a^2}{r^2} \right) \right]$$

$$= 1 - \frac{a^4}{r^4} + \frac{2a^2}{r^2} \cos 2\theta$$

$$(7) \quad \psi = \frac{m}{2\pi} \tan^{-1} \left(\frac{y}{x-a} \right) + \frac{m}{2\pi} \tan^{-1} \left(\frac{y}{x+a} \right)$$

$$= \frac{m}{2\pi} \tan^{-1} \left(\frac{\frac{y}{x-a} + \frac{y}{x+a}}{1 - \frac{y^2}{x^2 - a^2}} \right)$$

$$= \frac{m}{2\pi} \tan^{-1} \left(\frac{2xy}{x^2 - y^2 - a^2} \right)$$

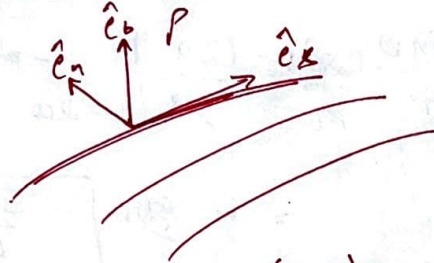


(8)

$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} + \frac{\nabla p}{\rho} = 0$$

EULER'S EQUATIONS

$$\underline{u} = u \hat{e}_s$$



$$\underline{u} \cdot \nabla = u \frac{\partial}{\partial s}$$

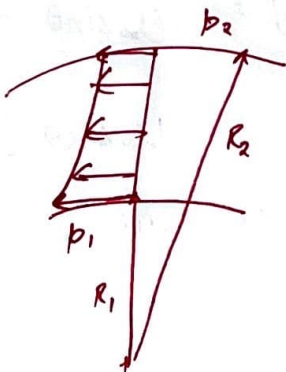
$$(\underline{u} \cdot \nabla) \underline{u} = u \frac{\partial}{\partial s} (u \hat{e}_s) = \hat{e}_s \frac{\partial}{\partial s} \left(\frac{u^2}{2} \right) + u^2 \frac{\partial \hat{e}_s}{\partial s}$$

$$\frac{\partial \hat{e}_s}{\partial s} = -\frac{\hat{e}_n}{R}$$

$R \rightarrow$ RADIUS OF CURVATURE

$$\Rightarrow \frac{\partial u}{\partial t} + \frac{\partial}{\partial s} \left(\frac{u^2}{2} \right) + \frac{1}{\rho} \frac{\partial p}{\partial s} = 0, \quad \frac{u^2}{R} = \frac{1}{\rho} \frac{\partial p}{\partial n}, \quad \frac{\partial p}{\partial \theta} = 0$$

(9)



$$u_\theta = \frac{C}{r}, \quad \frac{u_\theta^2}{r} = \frac{1}{\rho} \frac{\partial p}{\partial r} = \frac{C^2}{r^3}$$

$$\frac{\partial p}{\partial r} = \frac{\rho C^2}{r^3} \Rightarrow p_2 - p_1 = \rho \int_{R_1}^{R_2} \frac{C^2}{r^3} dr$$

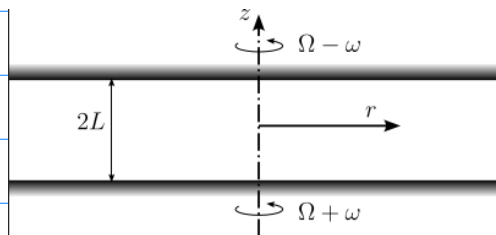
$$\Rightarrow C = \sqrt{\frac{2 \Delta p}{\rho} \left(\frac{1}{R_1^2} - \frac{1}{R_2^2} \right)} = \frac{\sqrt{\rho C^2}}{2} \left(\frac{1}{R_1^2} - \frac{1}{R_2^2} \right) = \Delta p$$

Flow RATE

$$\int_{R_1}^{R_2} u_\theta dr = \ln \left(\frac{R_2}{R_1} \right) \sqrt{\frac{2 \Delta p}{\rho} \left(\frac{1}{R_1^2} - \frac{1}{R_2^2} \right)}$$

$$a) \rho: (\underline{u} \cdot \nabla) u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r}$$

$$+ \gamma \left[\frac{\partial^2 u}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) - \frac{u}{r^2} \right]$$



using $u_r = r\omega U(z)$, $u_\theta = r\Omega + r\omega V(z)$, $u_z = \omega L W(z)$

$$p = \frac{1}{2} \rho \Omega^2 r^2 + \frac{1}{2} \rho \Omega \omega r^2 A + \rho \Omega \omega L^2 B(z), \quad z = \frac{z}{L}$$

$$\rho: \quad \cancel{\rho \omega^2 U^2} + \cancel{\rho \omega \Omega U'W} - \left(\cancel{r\Omega^2} + \cancel{\rho \omega^2 V^2} + \cancel{2\rho \Omega \omega V} \right) = -\cancel{\Omega^2 r} - \cancel{\Omega \omega A} + \gamma \frac{\rho \omega U''}{L^2}$$

$$\frac{\omega}{\Omega} \left(U^2 - V^2 + U'W \right) = 2V - A + \underbrace{\frac{\gamma}{L^2 \Omega}}_{\text{Ekman Number}} U''$$

ROSSBY NUMBER
 Ro

Ekman Number
(Ek)

$$\theta: \quad \frac{u_r u_\theta}{r} + u_r \frac{\partial u_\theta}{\partial r} + u_z \frac{\partial u_\theta}{\partial z} = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \gamma \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_\theta}{\partial r} \right) - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} \right]$$

$$r\omega U \left(\Omega + \omega V \right) + \omega L W \left(\frac{r\omega V'}{L} \right) + r\omega U \left(\frac{r\Omega + r\omega V}{r} \right) = \gamma \left[\frac{\Omega}{r} + \frac{\omega V'}{r} - \left(\frac{\Omega}{r} + \frac{\omega V'}{r} \right) + \frac{V'' r \omega}{L^2} \right]$$

$$Ro \left[UV + WV' \right] = -2U + Ek V''$$

$$z: \quad u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \gamma \left[\frac{\partial^2 u_z}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2} \right]$$

$$(L\omega)^2 \frac{W W'}{L} = -\frac{1}{P} \left(P \Sigma \omega L^2 \frac{B'}{L} \right) + \gamma \frac{W'' L \omega}{L^2}$$

$$\Rightarrow R_0(W W') = -B' + E_K W''$$

CONTINUITY: $\frac{1}{\eta} \frac{\partial}{\partial \eta} (\eta u_\eta) + \frac{1}{\eta} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$

$$\frac{2\omega \eta U}{\eta} + \frac{\omega L W'}{L} = 0 \Rightarrow 2U + W' = 0$$

b) NO SLIP: $U(\pm 1) = 0$, $V(\pm 1) = \mp 1$, $W(\pm 1) = 0$
 & HARD WALL

c) R_0 MULTIPLIES NONLINEAR TERMS. FOR $R_0 \ll 1$, AT LEADING ORDER

$$A - 2V = E_K U''$$

$$2U = E_K V''$$

$$W' = -2U$$

$$B' = E_K W''$$

d) LET $U(z) = U_0(z) + E_K U_1(z)$ & SAME FOR $V(z)$, $W(z)$
 A , $B(z)$

IN $\lim E_K \rightarrow 0$, OUTER SOLUTION

$$U_0 = 0, U_1 = 0$$

$$V_0 = \frac{A_0}{2}, V_1 = \frac{A_1}{2}, W_0, W_1 \text{ ARBITRARY CONSTANTS}$$

NO NON TRIVIAL SOLUTION WHICH SATISFIES B.C.s

THE CORR. DIMENSIONAL SOLUTION IS SIMPLY

$$u_\eta = 0, u_\theta = (\Sigma \pm \omega \eta), p = \frac{1}{2} P \Sigma^2 \eta^2 \pm P \Sigma \omega \eta^3, u_z = 0$$

c) LET $\eta_+ = \frac{1-z}{E_k^\beta}$, $\eta_- = \frac{1+z}{E_k^\beta}$

$$\frac{d}{dz} = \mp E_k^\beta \frac{d}{d\eta_\pm}$$

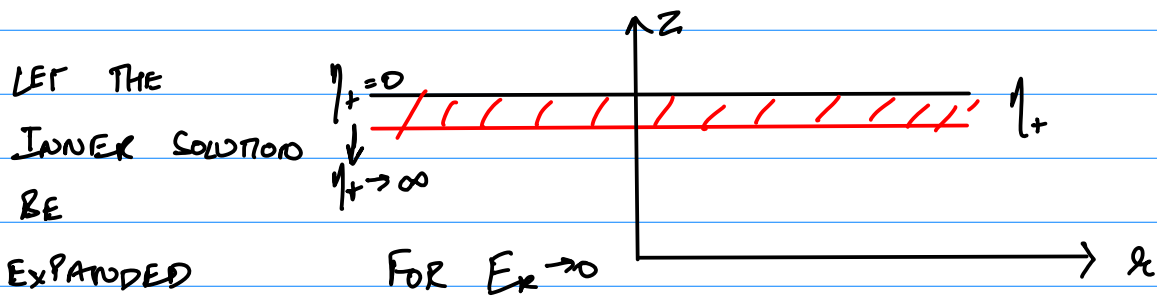
$$\frac{d^2}{dz^2} = E_k^{-2\beta} \frac{d^2}{d\eta_\pm^2}$$

LET

$$U(z) = U_I(\eta_\pm) \text{ \& so on}$$

$$E_k^{1-2\beta} U_I'' = A_0 - 2V_I, \quad E_k^{1-2\beta} V_I'' = 2U_I, \quad E_k^{-\beta} W_I' = \mp 2U_I$$

η_+ IS ZOOMED IN REGION NEAR $z=1$, SIMILARLY η_- FOR $z=-1$



$$U_I = U_I^{(0)} + \sqrt{E_k} U_I^{(1)}$$

WE GET AND FOR OTHER VARIABLES

$$V_I^{(0)''''} + 4V_I^{(0)} = 2A^{(0)}, \quad V_I^{(0)} = -1, \quad V_I^{(0)'} = 0$$

GENERAL SOLUTION OF

$$y'''' + 4y = 2K$$

$$y = Ae^{\sqrt{2}ix} + Be^{-\sqrt{2}ix} + Ce^{i\sqrt{2}ix} + De^{-i\sqrt{2}ix} + E$$

$$y'''' = -4(Ae^{\sqrt{2}ix} + Be^{-\sqrt{2}ix} + Ce^{i\sqrt{2}ix} + De^{-i\sqrt{2}ix})$$

$$y'''' + 4y = 4E = 2K \Rightarrow E = \frac{K}{2}$$

$$y(0) = -1 \Rightarrow A + B + C + D = -1 - \frac{K}{2}$$

$$\sqrt{2}i x = \frac{1+i}{\sqrt{2}} x$$

$$y = A e^{(1+i)x} + B e^{-(1+i)x} + C e^{i(1+i)x} + D e^{-i(1+i)x} + \frac{K}{2}$$

$$y''(0) = (1+i)^2 (A+B) - (1+i)^2 (C+D) = 0$$

$$\Rightarrow A+B = C+D$$

$$y = A (\sinh x + \cosh x) (\cos x + i \sin x) + B (\cosh x - \sinh x) (\cos x - i \sin x) + C (\cosh x - \sinh x) (\cos x + i \sin x) + D (\cosh x + \sinh x) (\cos x - i \sin x) + \frac{K}{2}$$

$$\begin{aligned} & iA (\sinh x \sin x) + iB \sinh x \sin x \\ & - iC \sinh x \sin x - iD \sinh x \sin x \quad \text{CANCELS OUT} \\ & A (\cosh x \cos x) + B \cosh x \cos x + C \cosh x \cos x + D \cosh x \cos x \\ & = 2(A+B) \cosh x \cos x = -\left(1 + \frac{K}{2}\right) \cosh x \cos x \end{aligned}$$

$$\begin{aligned} & A \sinh x \cos x - B \sinh x \cos x \\ & - C \sinh x \cos x + D \sinh x \cos x \end{aligned} =$$

$$\begin{aligned} \text{LET } A+B+C+D &= \left(1 + \frac{K}{2}\right) \\ A-B+D-C &= 2(A+D) - (A+B+C+D) \end{aligned}$$

$$= \left[1 + \left(1 + \frac{K}{2}\right)\right] \sinh x \cos x$$

$$\begin{aligned} & iA \cosh x \sin x - iB \cosh x \sin x + iC \cosh x \sin x - iD \cosh x \sin x \\ & = i(A-B) \cosh x \sin x + i(C-D) \cosh x \sin x \\ & = C_2 \cosh x \sin x \end{aligned}$$

$$\frac{K}{2} = \left(1 + \frac{K}{2}\right) e^{-x} \cos x + C_1 \sinh x \cos x + C_2 \cosh x \sin x$$

$$\Rightarrow V_I^{(0)} = \frac{A^{(0)}}{2} - \left(1 + \frac{A^{(0)}}{2}\right) e^{-\eta/4} \cos \eta_r + C_1 \sinh(\eta_r) \cos(\eta_r) + C_2 \cosh(\eta_r) \sin(\eta_r)$$

f) $V_I^{(0)}|_{\eta_+ \rightarrow \infty} = \text{OUTER SOLUTION FOR } Z \rightarrow 1 = \frac{A_0}{2}$

SINCE OUTER SOLUTIONS ARE CONSTANT $\Rightarrow C_1 = C_2 = 0$

$$U_I^{(0)} = \frac{1}{2} V_I^{(0)''} \Rightarrow \sim e^{-\eta_+} = 0 \text{ FOR } \eta_+ \rightarrow \infty$$

$$W_I^{(0)} = 0, \quad W_I^{(0)'} = -U_I^{(0)} \quad \left(E_k^{\frac{1}{2}} W_I' = \mp 2U_I \right)$$

$$U_I^{(0)} = -\frac{1}{2} \left(1 + \frac{A^{(0)}}{2} \right) 2e^{-\eta_+} \sin \eta_+ = \left(1 + \frac{A^{(0)}}{2} \right) e^{-\eta_+} \sin \eta_+$$

$$(e^{-x} \cos x)'' = (-e^{-x} \cos x - e^{-x} \sin x)' = (e^{-x} \cos x + e^{-x} \sin x + e^{-x} \sin x - e^{-x} \cos x)$$

$$W_I^{(1)} = -\left(1 + \frac{A^{(0)}}{2} \right) \int e^{-\eta_+} \sin \eta_+ d\eta_+ = \frac{1}{2} \left(1 + \frac{A^{(0)}}{2} \right) [\sin \eta_+ + \cos \eta_+] e^{-\eta_+} + C$$

$$W_I^{(1)}(\eta_+ = 0) = 0 \Rightarrow C = -\frac{1}{2} \left(1 + \frac{A^{(0)}}{2} \right) e$$

$$\Rightarrow W_I^{(1)} = \frac{1}{2} \left(1 + \frac{A^{(0)}}{2} \right) \left[e^{-\eta_+} (\sin \eta_+ + \cos \eta_+) - 1 \right]$$

FOR $\eta_+ \rightarrow \infty \quad W_I^{(1)} \rightarrow -\frac{1}{2} \left(1 + \frac{A^{(0)}}{2} \right)$

FROM η_- SOLUTION, WE FIND $W_I^{(1)} = -\frac{1}{2} \left(1 - \frac{A^{(0)}}{2} \right)$

FOR OUTER SOLUTION TO BE CONSTANT $W_I^{(1)} = -\frac{1}{A^{(0)} = 0}$

g) COMPOSITE SOLUTION IS SIMPLY

$$U = e^{-\eta_-} \sin \eta_- - e^{-\eta_+} \sin \eta_+, \quad V = e^{-\eta_-} \cos \eta_- - e^{-\eta_+} \cos \eta_+$$

$$W = \sqrt{E_k} \left[-1 + e^{-\eta_+} (\sin \eta_+ + \cos \eta_+) + e^{-\eta_-} (\sin \eta_- + \cos \eta_-) \right]$$