

# Curvilinear Coordinates

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## ***B1. Cylindrical Polar Coordinates***

The coordinates are  $(R, \theta, x)$ , where  $\theta$  is the azimuthal angle (see Figure 3.1b, where  $\varphi$  is used instead of  $\theta$ ). The equations are presented assuming  $\psi$  is a scalar, and

$$\mathbf{u} = \mathbf{i}_R u_R + \mathbf{i}_\theta u_\theta + \mathbf{i}_x u_x,$$

where  $\mathbf{i}_R$ ,  $\mathbf{i}_\theta$ , and  $\mathbf{i}_x$  are the local unit vectors at a point.

*Gradient of a scalar*

$$\nabla \psi = \mathbf{i}_R \frac{\partial \psi}{\partial R} + \frac{\mathbf{i}_\theta}{R} \frac{\partial \psi}{\partial \theta} + \mathbf{i}_x \frac{\partial \psi}{\partial x}.$$

*Laplacian of a scalar*

$$\nabla^2 \psi = \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \psi}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial x^2}.$$

*Divergence of a vector*

$$\nabla \cdot \mathbf{u} = \frac{1}{R} \frac{\partial (R u_R)}{\partial R} + \frac{1}{R} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_x}{\partial x}.$$

*Curl of a vector*

$$\nabla \times \mathbf{u} = \mathbf{i}_R \left( \frac{1}{R} \frac{\partial u_x}{\partial \theta} - \frac{\partial u_\theta}{\partial x} \right) + \mathbf{i}_\theta \left( \frac{\partial u_R}{\partial x} - \frac{\partial u_x}{\partial R} \right) + \mathbf{i}_x \left[ \frac{1}{R} \frac{\partial (R u_\theta)}{\partial R} - \frac{1}{R} \frac{\partial u_R}{\partial \theta} \right].$$

*Laplacian of a vector*

$$\nabla^2 \mathbf{u} = \mathbf{i}_R \left( \nabla^2 u_R - \frac{u_R}{R^2} - \frac{2}{R^2} \frac{\partial u_\theta}{\partial \theta} \right) + \mathbf{i}_\theta \left( \nabla^2 u_\theta + \frac{2}{R^2} \frac{\partial u_R}{\partial \theta} - \frac{u_\theta}{R^2} \right) + \mathbf{i}_x \nabla^2 u_x.$$

Strain rate and viscous stress (for incompressible form  $\sigma_{ij} = 2\mu e_{ij}$ )

$$\begin{aligned} e_{RR} &= \frac{\partial u_R}{\partial R} = \frac{1}{2\mu} \sigma_{RR}, \\ e_{\theta\theta} &= \frac{1}{R} \frac{\partial u_\theta}{\partial \theta} + \frac{u_R}{R} = \frac{1}{2\mu} \sigma_{\theta\theta}, \\ e_{xx} &= \frac{\partial u_x}{\partial x} = \frac{1}{2\mu} \sigma_{xx}, \\ e_{R\theta} &= \frac{R}{2} \frac{\partial}{\partial R} \left( \frac{u_\theta}{R} \right) + \frac{1}{2R} \frac{\partial u_R}{\partial \theta} = \frac{1}{2\mu} \sigma_{R\theta}, \\ e_{\theta x} &= \frac{1}{2R} \frac{\partial u_x}{\partial \theta} + \frac{1}{2} \frac{\partial u_\theta}{\partial x} = \frac{1}{2\mu} \sigma_{\theta x}, \\ e_{xR} &= \frac{1}{2} \frac{\partial u_R}{\partial x} + \frac{1}{2} \frac{\partial u_x}{\partial R} = \frac{1}{2\mu} \sigma_{xR}. \end{aligned}$$

Vorticity ( $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ )

$$\begin{aligned} \omega_R &= \frac{1}{R} \frac{\partial u_x}{\partial \theta} - \frac{\partial u_\theta}{\partial x}, \\ \omega_\theta &= \frac{\partial u_R}{\partial x} - \frac{\partial u_x}{\partial R}, \\ \omega_x &= \frac{1}{R} \frac{\partial}{\partial R} (Ru_\theta) - \frac{1}{R} \frac{\partial u_R}{\partial \theta}. \end{aligned}$$

Equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (\rho Ru_R) + \frac{1}{R} \frac{\partial}{\partial \theta} (\rho u_\theta) + \frac{\partial}{\partial x} (\rho u_x) = 0.$$

Navier–Stokes equations with constant  $\rho$  and  $\nu$ , and no body force

$$\begin{aligned} \frac{\partial u_R}{\partial t} + (\mathbf{u} \cdot \nabla) u_R - \frac{u_\theta^2}{R} &= -\frac{1}{\rho} \frac{\partial p}{\partial R} + \nu \left( \nabla^2 u_R - \frac{u_R}{R^2} - \frac{2}{R^2} \frac{\partial u_\theta}{\partial \theta} \right), \\ \frac{\partial u_\theta}{\partial t} + (\mathbf{u} \cdot \nabla) u_\theta + \frac{u_R u_\theta}{R} &= -\frac{1}{\rho R} \frac{\partial p}{\partial \theta} + \nu \left( \nabla^2 u_\theta + \frac{2}{R^2} \frac{\partial u_R}{\partial \theta} - \frac{u_\theta}{R^2} \right), \\ \frac{\partial u_x}{\partial t} + (\mathbf{u} \cdot \nabla) u_x &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 u_x, \end{aligned}$$

where

$$\begin{aligned} \mathbf{u} \cdot \nabla &= u_R \frac{\partial}{\partial R} + \frac{u_\theta}{R} \frac{\partial}{\partial \theta} + u_x \frac{\partial}{\partial x}, \\ \nabla^2 &= \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial x^2}. \end{aligned}$$

## B2. Plane Polar Coordinates

The plane polar coordinates are  $(r, \theta)$ , where  $r$  is the distance from the origin (Figure 3.1a). The equations for plane polar coordinates can be obtained from those of the cylindrical coordinates presented in Section B1, replacing  $R$  by  $r$  and suppressing all components and derivatives in the axial direction  $x$ . Some of the expressions are repeated here because of their frequent occurrence.

*Strain rate and viscous stress* (for incompressible form  $\sigma_{ij} = 2\mu e_{ij}$ )

$$\begin{aligned} e_{rr} &= \frac{\partial u_r}{\partial r} = \frac{1}{2\mu} \sigma_{rr}, \\ e_{\theta\theta} &= \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = \frac{1}{2\mu} \sigma_{\theta\theta}, \\ e_{r\theta} &= \frac{r}{2} \frac{\partial}{\partial r} \left( \frac{u_\theta}{r} \right) + \frac{1}{2r} \frac{\partial u_r}{\partial \theta} = \frac{1}{2\mu} \sigma_{r\theta}. \end{aligned}$$

*Vorticity*

$$\omega_z = \frac{1}{r} \frac{\partial}{\partial r} (ru_\theta) - \frac{1}{r} \frac{\partial u_r}{\partial \theta}.$$

*Equation of continuity*

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (\rho r u_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho u_\theta) = 0.$$

*Navier–Stokes equations with constant  $\rho$  and  $\nu$ , and no body force*

$$\begin{aligned} \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} &= -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left( \nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right), \\ \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} &= -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \nu \left( \nabla^2 u_\theta + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right), \end{aligned}$$

where

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.$$

## B3. Spherical Polar Coordinates

The spherical polar coordinates used are  $(r, \theta, \varphi)$ , where  $\varphi$  is the azimuthal angle (Figure 3.1c). Equations are presented assuming  $\psi$  is a scalar, and

$$\mathbf{u} = \mathbf{i}_r u_r + \mathbf{i}_\theta u_\theta + \mathbf{i}_\varphi u_\varphi,$$

where  $\mathbf{i}_r$ ,  $\mathbf{i}_\theta$ , and  $\mathbf{i}_\varphi$  are the local unit vectors at a point.

*Gradient of a scalar*

$$\nabla \psi = \mathbf{i}_r \frac{\partial \psi}{\partial r} + \mathbf{i}_\theta \frac{1}{r} \frac{\partial \psi}{\partial \theta} + \mathbf{i}_\varphi \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \varphi}.$$

*Laplacian of a scalar*

$$\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \varphi^2}.$$

*Divergence of a vector*

$$\nabla \cdot \mathbf{u} = \frac{1}{r^2} \frac{\partial (r^2 u_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (u_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial u_\varphi}{\partial \varphi}.$$

*Curl of a vector*

$$\begin{aligned} \nabla \times \mathbf{u} = & \frac{\mathbf{i}_r}{r \sin \theta} \left[ \frac{\partial (u_\varphi \sin \theta)}{\partial \theta} - \frac{\partial u_\theta}{\partial \varphi} \right] + \frac{\mathbf{i}_\theta}{r} \left[ \frac{1}{\sin \theta} \frac{\partial u_r}{\partial \varphi} - \frac{\partial (r u_\varphi)}{\partial r} \right] \\ & + \frac{\mathbf{i}_\varphi}{r} \left[ \frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta} \right]. \end{aligned}$$

*Laplacian of a vector*

$$\begin{aligned} \nabla^2 \mathbf{u} = & \mathbf{i}_r \left[ \nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2 \sin \theta} \frac{\partial (u_\theta \sin \theta)}{\partial \theta} - \frac{2}{r^2 \sin \theta} \frac{\partial u_\varphi}{\partial \varphi} \right] \\ & + \mathbf{i}_\theta \left[ \nabla^2 u_\theta + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2 \sin^2 \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial u_\varphi}{\partial \varphi} \right] \\ & + \mathbf{i}_\varphi \left[ \nabla^2 u_\varphi + \frac{2}{r^2 \sin \theta} \frac{\partial u_r}{\partial \varphi} + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial u_\theta}{\partial \varphi} - \frac{u_\varphi}{r^2 \sin^2 \theta} \right]. \end{aligned}$$

*Strain rate and viscous stress (for incompressible form  $\sigma_{ij} = 2\mu e_{ij}$ )*

$$\begin{aligned} e_{rr} &= \frac{\partial u_r}{\partial r} = \frac{1}{2\mu} \sigma_{rr}, \\ e_{\theta\theta} &= \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = \frac{1}{2\mu} \sigma_{\theta\theta}, \\ e_{\varphi\varphi} &= \frac{1}{r \sin \theta} \frac{\partial u_\varphi}{\partial \varphi} + \frac{u_r}{r} + \frac{u_\theta \cot \theta}{r} = \frac{1}{2\mu} \sigma_{\varphi\varphi}, \\ e_{\theta\varphi} &= \frac{\sin \theta}{2r} \frac{\partial}{\partial \theta} \left( \frac{u_\varphi}{\sin \theta} \right) + \frac{1}{2r \sin \theta} \frac{\partial u_\theta}{\partial \varphi} = \frac{1}{2\mu} \sigma_{\theta\varphi}, \\ e_{\varphi r} &= \frac{1}{2r \sin \theta} \frac{\partial u_r}{\partial \varphi} + \frac{r}{2} \frac{\partial}{\partial r} \left( \frac{u_\varphi}{r} \right) = \frac{1}{2\mu} \sigma_{\varphi r}, \\ e_{r\theta} &= \frac{r}{2} \frac{\partial}{\partial r} \left( \frac{u_\theta}{r} \right) + \frac{1}{2r} \frac{\partial u_r}{\partial \theta} = \frac{1}{2\mu} \sigma_{r\theta}. \end{aligned}$$

Vorticity

$$\begin{aligned}\omega_r &= \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (u_\varphi \sin \theta) - \frac{\partial u_\theta}{\partial \varphi} \right], \\ \omega_\theta &= \frac{1}{r} \left[ \frac{1}{\sin \theta} \frac{\partial u_r}{\partial \varphi} - \frac{\partial (ru_\varphi)}{\partial r} \right], \\ \omega_\varphi &= \frac{1}{r} \left[ \frac{\partial}{\partial r} (ru_\theta) - \frac{\partial u_r}{\partial \theta} \right].\end{aligned}$$

Equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (\rho r^2 u_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\rho u_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} (\rho u_\varphi) = 0.$$

Navier–Stokes equations with constant  $\rho$  and  $\nu$ , and no body force

$$\begin{aligned}\frac{\partial u_r}{\partial t} + (\mathbf{u} \cdot \nabla) u_r - \frac{u_\theta^2 + u_\varphi^2}{r} \\ = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left[ \nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2 \sin \theta} \frac{\partial (u_\theta \sin \theta)}{\partial \theta} - \frac{2}{r^2 \sin \theta} \frac{\partial u_\varphi}{\partial \varphi} \right], \\ \frac{\partial u_\theta}{\partial t} + (\mathbf{u} \cdot \nabla) u_\theta + \frac{u_r u_\theta}{r} - \frac{u_\varphi^2 \cot \theta}{r} \\ = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \nu \left[ \nabla^2 u_\theta + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2 \sin^2 \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial u_\varphi}{\partial \varphi} \right], \\ \frac{\partial u_\varphi}{\partial t} + (\mathbf{u} \cdot \nabla) u_\varphi + \frac{u_\varphi u_r}{r} + \frac{u_\theta u_\varphi \cot \theta}{r} \\ = -\frac{1}{\rho r \sin \theta} \frac{\partial p}{\partial \varphi} + \nu \left[ \nabla^2 u_\varphi + \frac{2}{r^2 \sin \theta} \frac{\partial u_r}{\partial \varphi} + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial u_\theta}{\partial \varphi} - \frac{u_\varphi}{r^2 \sin^2 \theta} \right],\end{aligned}$$

where

$$\begin{aligned}\mathbf{u} \cdot \nabla &= u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + \frac{u_\varphi}{r \sin \theta} \frac{\partial}{\partial \varphi}, \\ \nabla^2 &= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}.\end{aligned}$$

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