

Tutorial-1 : Vector calculus, index notation, tensor algebra and calculus

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1. Using divergence theorem, show that,

$$V = \frac{1}{3} \int_S \underline{n} \cdot \underline{r} dS, \quad (1)$$

where S is the surface enclosing the volume V . Using above equation, find the volume of,

- a rectangular parallelopiped with sides a, b and c
 - a right circular cone with height H and base radius R
 - a sphere of radius R
2. Evaluate the surface integral $\iint_S \underline{F} \cdot \underline{n} dS$ where $\underline{F} = e^x \hat{i} + e^y \hat{j} + e^z \hat{k}$ and S is the surface of the cube defined by $|x| \leq 1, |y| \leq 1$, and $|z| \leq 1$.
3. Show that area enclosed by a curve C is

$$A = \frac{1}{2} \oint_C (\underline{r} \times \underline{t}) \cdot \hat{k} dl. \quad (2)$$

4. A piston is vibrating in a resonating tube of cross-sectional area S . The location of the piston is given by $x_L = L + \ell \sin \omega t$. The other end of the tube is fixed at $x = 0$. Density of the gas is varying with a different rate as $\rho = \rho_0 + r \cos \omega t$ which is imposed by some external agent. Using Leibniz formula, compute the rate of change of mass of the cylinder.
5. Consider a 2D flow in a layer whose thickness is $\delta(x) = \sqrt{x}$ and flow speed is $u(y) = ky$. Calculate the volume flux given by $\int_0^{\delta(x)} u dy$ and its rate of change with x .
6. Suppose a $\partial f / \partial x = e^x$ is known for a real valued function $f(x, y)$
- Show that you can write $f(x, y) = \int \frac{\partial f}{\partial x} dx + g(y) + C$ for a finite valued smooth f with x and y as independent variables.
 - Solve for $f(x, y)$ in the above equation given that $f(x = 0, y) = 5 + \sin(y)$.
7. For a function $z(x, y)$, show the following identity,

$$\left. \frac{\partial z}{\partial x} \right|_y \left. \frac{\partial x}{\partial y} \right|_z \left. \frac{\partial y}{\partial z} \right|_x = -1. \quad (3)$$

8. Consider a real square matrix A and its decomposition in diagonal and deviatoric parts, D and T respectively.
- Show that if A is symmetric, T is symmetric.
 - Consider that these matrices are 3×3 . If all diagonal components of D are p , then compute $\text{tr}(T)$ in terms of $\text{tr}(A)$.

- Consider a column vector v and product Av . Discuss the effect of D and T on v separately.
9. A coordinate system $\bar{e}$ is obtained from the e system by a counter-clockwise rotation of 90° about the x_3 axis. The e^* coordinate system is obtained from the e system by a reflection of the x_3 axis about the $x_1 - x_2$ plane.
- Sketch the e , $\bar{e}$ and the e^* coordinate systems.
 - Write $\bar{e}_i$ and e_i^* in terms of e_i .
 - Determine the direction cosines, $\bar{a}_{ij}$ ($\equiv e_i \cdot \bar{e}_j$) and a_{ij}^* ($\equiv e_i \cdot e_j^*$).
 - Verify that $\bar{a}_{ik}\bar{a}_{jk} = \delta_{ij}$ and $a_{ik}^*a_{jk}^* = \delta_{ij}$
 - Determine the determinants of $\bar{a}_{ij}$ and a_{ij}^* .
10. With ϕ being a scalar, $\underline{u}$ a vector, and $\underline{\omega} \equiv \underline{\nabla} \times \underline{u}$, show using the index notation,
- $\underline{\nabla} \cdot \underline{\omega} = 0$
 - $\underline{\nabla} \times (\underline{\nabla} \phi) = 0$
 - $\underline{\nabla} \times (\underline{\nabla} \times \underline{u}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{u}) - \underline{\nabla}^2 \underline{u}$
 - $\underline{u} \times \underline{\omega} = \frac{1}{2} (\nabla(\underline{u} \cdot \underline{u}) - (\underline{u} \cdot \underline{\nabla}) \underline{u})$
11. Let $r = \sqrt{(x_k x_k)}$ denote the distance of a point x from the origin. Using the index notation find:
- $\underline{\nabla} r = \partial r / \partial x_i$.
 - $\underline{\nabla} \left(\frac{1}{r} \right) = \partial / \partial x_i \left(\frac{1}{r} \right)$.
 - $\nabla^2 \left(\frac{1}{r} \right) = \partial^2 / \partial x_i \partial x_i \left(\frac{1}{r} \right)$.
 - $\underline{\nabla} \times \underline{r} = \epsilon_{ijk} \frac{\partial r_k}{\partial x_j}$
12. Show that if S_{ij} is a symmetric second order tensor and T_{ij} has both symmetric and anti-symmetric components T_{ij}^s and T_{ij}^a respectively,
- $S_{ij}T_{ij}^a = 0$
 - $S_{ij}T_{ij} = S_{ij}T_{ij}^s$
13. Consider a surface with normal unit vector $\hat{n}$. Surface gradient of a scalar ϕ is defined as,

$$\underline{\nabla}_s \phi = \underline{\nabla} \phi - (\hat{n} \cdot \underline{\nabla} \phi) \hat{n} \quad (4)$$

Show that $\underline{\nabla}_s \phi \cdot \hat{n} = 0$ using index notation.

14. To define force between 2 particles, each at locations $\underline{x}$ and $\underline{y}$, a potential can be defined as $V(r) = \frac{1}{4}r^4$ where $r^2 = (\underline{x}_i - \underline{y}_i)(\underline{x}_i - \underline{y}_i)$ is the square of the distance between the particles. Force on each particle is given as $-\frac{\partial V}{\partial \underline{x}}$ and $-\frac{\partial V}{\partial \underline{y}}$. Simplify these expressions.