

Tutorial 3 : Flow visualization, fluid deformation, material derivative (Answers)

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September 11, 2026

1. The surface temperature of a lake on a winter day changes from one location to another as a function of x and y as $T(x, y) = 250 + 10xy$. If you attach a thermometer to a boat and take a path through the lake given by $x_b = t^2$ and $y_b = 2t$ where t is time, find the rate of change of temperature as recorded by the thermometer.

Solution:-

$$T(x, y) = 250 + 10xy$$

$$\frac{dT}{dt} = \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt} = 10y \frac{dx}{dt} + 10x \frac{dy}{dt}$$

$$\left. \frac{dT}{dt} \right|_{therm} = 20t(2t) + 10t^2(2) = 60t^2$$

$$\left. \frac{dT}{dt} \right|_{therm} = 60t^2$$

2. Consider a square shaped fluid particle in the boundary layer as shown in figure 1. The velocity field inside the boundary layer is given by $u(x, y)$. Assume uni-directional flow.

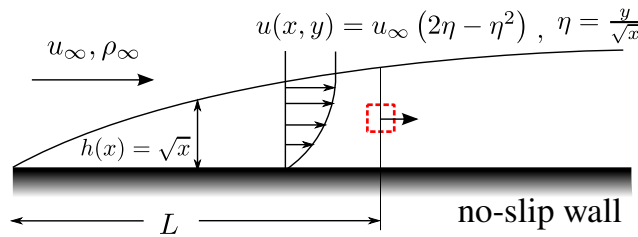


Figure 1: Deformation of a fluid particle in a boundary layer

- (a) Initial shape of the fluid parcel at $t = 0$ is shown. Sketch the shape of the red shaped fluid parcel after time δt .
- (b) Calculate the velocity gradient $\nabla \vec{v}$.
- (c) Decompose the velocity gradient in symmetric and anti-symmetric components.

Solution:- (a)

(b)

$$\nabla \vec{v} = \frac{\partial U_j}{\partial x_i} e_i e_j$$

Assume uni-directional flow,

$$\nabla \vec{v} = \begin{bmatrix} 0 & 0 \\ \frac{\partial u(x, y)}{\partial y} & 0 \end{bmatrix} \quad (0.0.1)$$

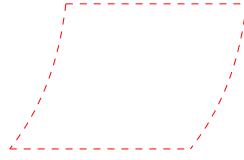


Figure 2: close view of fluid parcel at δt

$$\nabla \vec{v} = \begin{bmatrix} 0 & 0 \\ u_{\infty} \left(\frac{2}{\sqrt{x}} - \frac{2y}{x} \right) & 0 \end{bmatrix} \quad (0.0.2)$$

(c) $\nabla \vec{v}$ can be separated into symmetric and an antisymmetric components

$$\frac{\partial U_j}{\partial x_i} = \frac{1}{2} \left(\frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_j} \right) + \frac{1}{2} \left(\frac{\partial U_j}{\partial x_i} - \frac{\partial U_i}{\partial x_j} \right)$$

$$\partial_i v_j = S_{ij} + A_{ij}$$

3. An oscillating flow exists inside a planar channel of length $\ell = 0.1$ m and height $h = 1$ cm as shown in the figure below. The flow is uni-directional, and is given by

$$u(x, y, t) = 0.1 \sin(\omega t - kx) (1 - \cosh(10y) / \cosh(10h)),$$

where $k = 2\pi/\ell$, $\omega = 300$ rad/s, and origin is at the center of the channel. For the three fluid particles shown in the figure,

- Compute their acceleration as they pass through the coordinates shown.
- If the fluid is heated inside the channel and maintained at a spatially varying temperature $T = 300 + 50x$ K, calculate the rate of change of temperature of the fluid particles as they pass through the coordinates shown.

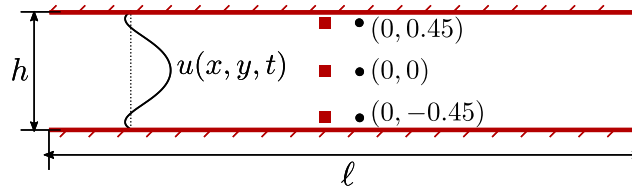


Figure 3: Oscillating flow in a channel

Solution:- (a)

$$\frac{du(x, y, t)}{dt} = \omega \cot(\omega t - kx) u(x, y, t) - k \cot(\omega t - kx) u^2(x, y, t)$$

(b)

$$\frac{dT(x, y, t)}{dt} = 50u(x, y, t)$$

4. A velocity field is given as $\vec{v} = Ax\hat{i} - Ay\hat{j}$, x and y are in meters and $A = 0.3 \text{ s}^{-1}$. Find,

- Equation of the streamline in $x - y$ plane.
- Velocity of the particle at point $(2, 8)$.

- (c) Position at $t = 6$ s of the particle located at $(2, 8)$ at $t = 0$.
- (d) Velocity of the particle in the above part at $t = 6$.
- (e) Equation of the particle in the above part starting at $(2, 8)$ at $t = 0$.

Solution:-

- (a) Given that $\vec{v} = Ax\hat{i} - Ay\hat{j}$

The streamline equation is

$$\frac{dx}{Ax} = \frac{dy}{-Ay} \quad (0.0.3)$$

solving the above equation we get,

$$\ln(x) + \ln(y) = \ln(c) \quad (0.0.4)$$

Hence, the equation of the streamline,

$$xy = c$$

- (b) At $(2,8)$,

$$\vec{v} = 0.3x\hat{i} - 0.3y\hat{j}$$

$$\vec{V} = 0.6\hat{i} - 2.4\hat{j}$$

The streamline passes through $(2,8)$ hence the equation of streamline becomes $xy=16$.

- (c) At $t = 0$ the position is $(2,8)$

$$(12.099, 1.322)m.$$

- (d)

$$\vec{v} = (3.63\hat{i} - 0.396\hat{j})m/s$$

- (e)

$$xy = 16$$

5. In a 2D flow, the velocity field is given by $\vec{v} = y\hat{i} + x\hat{j}$.

- (a) Find the equation of streamline passing through $(0, 0)$ at $t = 5$ s.
- (b) Find the equation of pathline of a particle released from $(0, 0)$ at $t = 0$.

Solution:-

- (a)

$$x^2 = y^2$$

- (b) The particle at $(0,0)$ remains there. No pathline

6. The figure below shows pathlines of particles released at $(0, 0)$ at a gap of 1 second ($0 \leq t \leq 8$). The time indicated in the figure shows the time at which the particle was released.

- (a) Explain giving reasons if this flow is steady or unsteady
- (b) Sketch the figure and on it draw a qualitative pattern of the streakline of particles passing through $(0, 0)$ observed at $t = 6$ s.

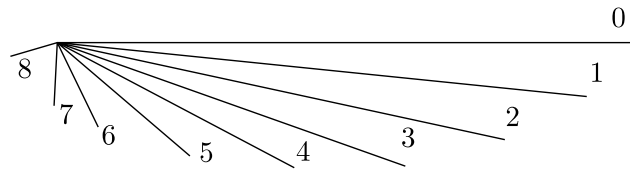


Figure 4: Pathlines

Solution:-

(a) The flow is unsteady . If the flow the flow had been steady each particle would have traced the same path as the first one. The change in trajectory of each particle indicates that the the path is time dependent. (b)

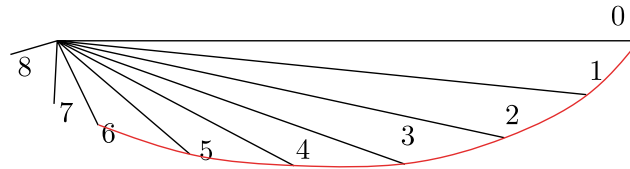


Figure 5: Streakline observed at $t = 6$

7. Sketch the streamlines of a plane flow (2D and bi-directional) for which the velocity field is given as,

$$u = 3(x^2 - y^2), \quad v = -6xy.$$

Solution:-

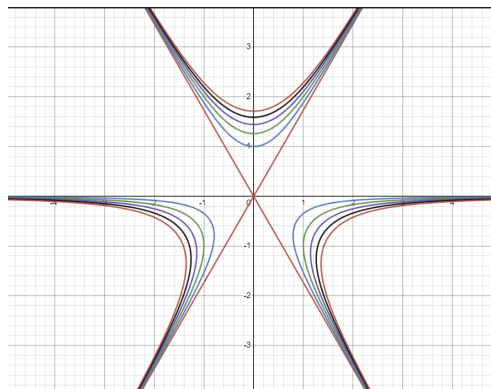


Figure 6: Streamlines of $y(y^2 - 3x^2) = K$

8. The velocity components in a plane flow are given by,

$$u = \frac{x}{1+t}, \quad v = \frac{2y}{2+t}.$$

(a) Find the equation of pathline of a fluid particle which was at $(1, 1)$ at $t = 0$.

(b) Find the equation of streaklines of all the particles observed at time t , passing through $(1, 1)$ at $0 < \tau < t$.

(c) Find the equations of streamlines passing through $(1, 1)$ observed at $t = 0$.

Describe pathlines and the streamlines of the flow.

Solution:-

(a)

$$x = (t + 1)$$

$$y = \left(\frac{t + 2}{2}\right)^2$$

(b)

$$x = \frac{1+t}{1+\tau}$$

$$y = \left(\frac{t+2}{\tau+2}\right)^2$$

(c)

$$x = y$$

9. For velocity field given in problem 2, calculate the acceleration of the fluid particle passing through (1, 1) at time t .

Solution:-

$$a_y|_{(1,1)} = \frac{2}{(t+2)^2}$$

10. Let a function $\psi(x, y, t)$ be defined for a planar flow such that,

$$\frac{\partial \psi}{\partial x} = -v, \quad \frac{\partial \psi}{\partial y} = u.$$

Show that the iso-contour lines of this function are parallel to the streamlines of the flow at all times. Iso-contours are the lines along which the function has a constant value.

Solution:-

$$d\psi = \left(\frac{\partial \psi}{\partial t}\right)dt + \left(\frac{\partial \psi}{\partial x}\right)dx + \left(\frac{\partial \psi}{\partial y}\right)dy$$

ψ is constant along its isocontours for all times, $d\psi = 0$

Therefore,

$$\begin{aligned} 0 &= \left(\frac{\partial \psi}{\partial x}\right)dx + \left(\frac{\partial \psi}{\partial y}\right)dy \\ 0 &= (-v)dx + (u)dy \end{aligned}$$

Hence, slope of the function ψ ,

$$\left(\frac{dy}{dx}\right)_\psi = \frac{v}{u}$$

As we know that the slope of any streamline,

$$\left(\frac{dy}{dx}\right)_{streamline} = \frac{v}{u}$$

These show the slopes of stream functions and the streamlines are same. Therefore, iso-contour lines of the stream function are parallel to the streamlines of the flow at all times

11. For a flow velocity field given in cylindrical coordinates,

$$\vec{v}(r, \theta, z, t) = v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z,$$

write the expression of the acceleration of a fluid particle. Refer to handout on derivatives in curvilinear coordinates for $\vec{v} \cdot \nabla$ in cylindrical coordinates.

Solution:- We know that,

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v},$$

Now,

$$\vec{v} \cdot \nabla = v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r} \frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z},$$

$$(\vec{v} \cdot \nabla) \vec{v} = \left(v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r} \frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z} \right) (v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z)$$

$$\begin{aligned} (\vec{v} \cdot \nabla) \vec{v} = & \left(v_r \frac{\partial v_r \hat{e}_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r \hat{e}_r}{\partial \theta} + v_z \frac{\partial v_r \hat{e}_r}{\partial z} \right) + \left(v_r \frac{\partial v_\theta \hat{e}_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta \hat{e}_\theta}{\partial \theta} + v_z \frac{\partial v_\theta \hat{e}_\theta}{\partial z} \right) + \\ & \left(v_r \frac{\partial v_z \hat{e}_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z \hat{e}_z}{\partial \theta} + v_z \frac{\partial v_z \hat{e}_z}{\partial z} \right) \end{aligned}$$

Only unit vectors $\hat{e}_r$ and $\hat{e}_\theta$ will change with respect to θ , i.e.,

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta, \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r \quad (0.0.5)$$

This gives you,

$$\begin{aligned} (\vec{v} \cdot \nabla) \vec{v} = & \left[v_r \frac{\partial v_r}{\partial r} \hat{e}_r + \frac{v_\theta}{r} \left(\frac{\partial v_r}{\partial \theta} \hat{e}_r + v_r \frac{\partial \hat{e}_r}{\partial \theta} \right) + v_z \frac{\partial v_r}{\partial z} \hat{e}_r \right] + \\ & \left[v_r \frac{\partial v_\theta}{\partial r} \hat{e}_\theta + \frac{v_\theta}{r} \left(\frac{\partial v_\theta}{\partial \theta} \hat{e}_\theta + v_\theta \frac{\partial \hat{e}_\theta}{\partial \theta} \right) + v_z \frac{\partial v_\theta}{\partial z} \hat{e}_\theta \right] + \\ & \left[v_r \frac{\partial v_z}{\partial r} \hat{e}_z + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} \hat{e}_z + v_z \frac{\partial v_z}{\partial z} \hat{e}_z \right] \end{aligned}$$

Using equation 0.0.5 and rearranging the RHS of the above equation, we will get,

$$\begin{aligned} (\vec{v} \cdot \nabla) \vec{v} = & \left[v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} \right] \hat{e}_r + \\ & \left[v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_\theta v_r}{r} \right] \hat{e}_\theta + \\ & \left[v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right] \hat{e}_z \end{aligned}$$

$$(\vec{v} \cdot \nabla) \vec{v} = \left[(\vec{v} \cdot \nabla) v_r - \frac{v_\theta^2}{r} \right] \hat{e}_r + \left[(\vec{v} \cdot \nabla) v_\theta + \frac{v_\theta v_r}{r} \right] \hat{e}_\theta + [(\vec{v} \cdot \nabla) v_z] \hat{e}_z$$

$$\frac{\partial \vec{v}}{\partial t} = \frac{\partial v_r}{\partial t} \hat{e}_r + \frac{\partial v_\theta}{\partial t} \hat{e}_\theta + \frac{\partial v_z}{\partial t} \hat{e}_z$$

Now the acceleration becomes,

$$\frac{d\vec{v}}{dt} = \left[\frac{\partial v_r}{\partial t} + (\vec{v} \cdot \nabla) v_r - \frac{v_\theta^2}{r} \right] \hat{e}_r + \left[\frac{\partial v_\theta}{\partial t} + (\vec{v} \cdot \nabla) v_\theta + \frac{v_\theta v_r}{r} \right] \hat{e}_\theta + \left[\frac{\partial v_z}{\partial t} + (\vec{v} \cdot \nabla) v_z \right] \hat{e}_z$$

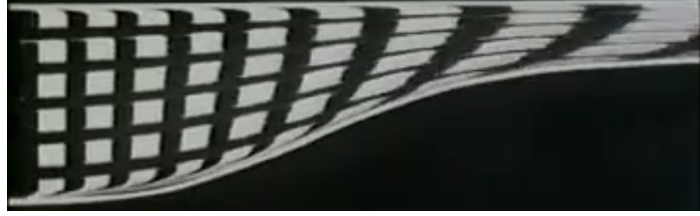


Figure 7: Hydrogen bubble flow visualization. Taken from the NCFMF movie on flow visualization.



12. In this Hydrogen bubble flow visualization of a flow in a converging channel, two primary flow field phenomenon can be observed. The bubble blobs change shape as they pass through the converging channel. Also, the blobs at the bottom tend to be lagging behind the blobs at the top. Using these observations, sketch the velocity profile at various x (horizontal) locations in the converging section. Can you also observe how the blobs close to the boundaries deform much more than the blobs in the middle? Why so?
Solution:- Due to no-slip, shear deformation rate is high close to the boundaries. So the blobs which have spent more time in the flow undergo much more shear deformation.
13. See the flow visualization video. Dye is injected in the cylinder and comes out of a small hole as the water flows past the cylinder. Notice the flow patterns downstream of the cylinder. Which line is traced by this dye? **Solution:-** Streakline