

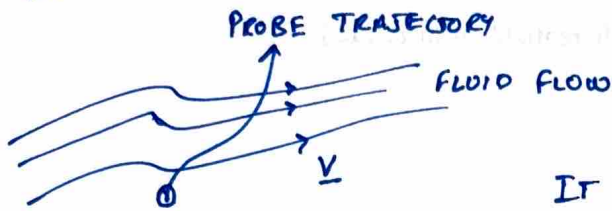
$$\textcircled{1} \quad \frac{\partial P}{\partial t} + \nabla \cdot (P \underline{v}) = 0 \Rightarrow \frac{\partial P}{\partial t} + P \nabla \cdot \underline{v} + \underline{v} \cdot \nabla P = 0 \Rightarrow \nabla \cdot \underline{v} = -\frac{1}{P} \left(\frac{\partial P}{\partial t} + \underline{v} \cdot \nabla P \right)$$

$$\Rightarrow \nabla \cdot \underline{v} = -\frac{1}{P} \frac{DP}{Dt} \quad \text{LET } P = \frac{1}{V} \rightarrow \text{SPECIFIC VOLUME} \Rightarrow \nabla \cdot \underline{v} = -\frac{1}{V} \frac{DV}{Dt}$$

$$\frac{1}{V} \frac{DV}{Dt} \rightarrow \text{BULK STRAIN RATE}$$

$$\frac{\Delta V}{V} \rightarrow \text{BULK STRAIN} = \frac{1}{V} \frac{DV}{Dt}$$

$\textcircled{2}$ a)



PROBE VELOCITY $\rightarrow \underline{w}$

IT MEASURES $\phi(\underline{y}(t), t)$

$$\text{WHERE } \frac{d\underline{y}}{dt} = \underline{w}$$

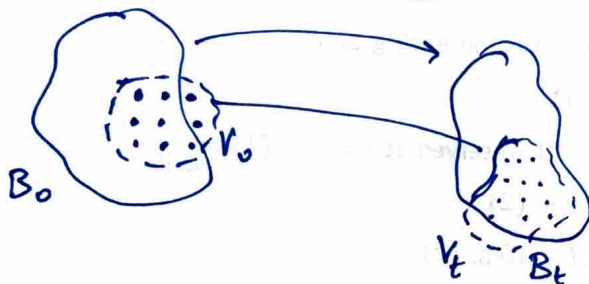
$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \frac{d\underline{y}}{dt} \cdot \frac{\partial \phi}{\partial \underline{y}}$$

$\rightarrow$ HOW DO WE KNOW THAT THIS $() \cdot ()$ PRODUCT COMES HERE?

INDEX NOTATION

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial y_i} \frac{\partial y_i}{\partial t} = \frac{\partial \phi}{\partial t} + \underline{w} \cdot \nabla \phi$$

b)



CONSIDER THE CONTROL VOLUME V_t AS A COLLECTION OF INFINITE NUMBER OF PROBES.

$$\int_{V_t} P \phi(\underline{y}, t) dV = \int_{V_0} P \phi(\underline{y}, t) J_v dV_0$$

$$\cancel{\frac{dV_t}{dt}} = \frac{dV_t}{dt} \quad J_v = \frac{dV_t}{dV_0}$$

$$\frac{d}{dt} \int_{V_t} P \phi(\underline{y}, t) dV = \frac{d}{dt} \int_{V_0} P \phi(\underline{y}, t) J_v dV_0$$

$$= \int_{V_0} \left[\frac{d}{dt} (P \phi) J_v + P \phi \frac{dJ_v}{dt} \right] dV_0$$

$$\text{FOR } \phi = 1 \quad \frac{d}{dt} \int_{V_t} P dV = \int_{V_0} \left(\frac{dP}{dt} J_v + P \frac{dJ_v}{dt} \right) dV_0$$

$$\frac{1}{J_v} \frac{dJ_v}{dt} = \nabla \cdot \underline{w} \quad (\text{CONSIDER "BULK STRAIN" IN CONTROL VOLUME RATE})$$

$$\Rightarrow \frac{d}{dt} \int_{V_t} P dV = \int_{V_0} \left(\frac{dP}{dt} J_v + P \nabla \cdot \underline{w} J_v \right) dV_0$$

$$\Rightarrow \frac{d}{dt} \int_{V_t} \rho dV = \int_{V_0} \left[\left(\frac{\partial \rho}{\partial t} + \underline{w} \cdot \nabla \rho \right) J_v + \rho \nabla \cdot \underline{w} J_v \right] dV$$

$$= \int_{V_0} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{w}) \right] J_v dV = \int_{V_t} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) \right) dV$$

FROM CONTINUITY, WE KNOW $\frac{\partial \rho}{\partial t} = - \nabla \cdot (\rho \underline{v})$

$$\Rightarrow \frac{d}{dt} \int_{V_t} \rho dV = - \int_{V_t} \nabla \cdot (\rho \underline{v} - \rho \underline{w}) dV = - \oint_{S_0} \rho (\underline{v} - \underline{w}) \cdot \hat{n} dS$$

$$3^a) \quad x_j \hat{e}_j \times \left(\frac{\partial (\rho u_i)}{\partial t} \hat{e}_i + \frac{\partial (\rho u_k u_i)}{\partial x_k} \hat{e}_i \right) = f_i \hat{e}_i + \frac{\partial (\tau_{ik})}{\partial x_k} \hat{e}_i$$

ASSUMING $\rightarrow$ CARTESIAN

$$\Rightarrow x_j \frac{\partial (\rho u_i)}{\partial t} \hat{e}_j \times \hat{e}_i + x_j \frac{\partial (\rho u_k u_i)}{\partial x_k} \hat{e}_j \times \hat{e}_i = x_j f_i \hat{e}_j \times \hat{e}_i$$

↑
EULERIAN COORDINATE

$$+ x_j \frac{\partial \tau_{ik}}{\partial x_k} \hat{e}_j \times \hat{e}_i$$

$$\Rightarrow \frac{\partial (\rho u_i x_j)}{\partial t} \epsilon_{ijl} \hat{e}_l + x_j \frac{\partial (\rho u_k u_i)}{\partial x_k} \epsilon_{ijl} \hat{e}_l = x_j f_i \epsilon_{ijl} \hat{e}_l$$

$$+ x_j \frac{\partial \tau_{ik}}{\partial x_k} \epsilon_{ijl} \hat{e}_l$$

b) REPLACING ϕ by $\epsilon_{ijk} x_j u_k$.

$$\int \frac{\partial}{\partial t} (\epsilon_{ijk} x_j u_k \rho) dV + \int \frac{\partial}{\partial x_k} (\rho u_k \epsilon_{ijk} x_j u_k) dV = \int \epsilon_{ijk} x_j f_k dV$$

$$+ \frac{\partial}{\partial x_k} \oint (\underline{t} \times \underline{I}) \cdot \hat{n} dS$$

$$\oint (\underline{t} \times \underline{I}) \cdot \hat{n} dS = \int_V \nabla \cdot (\underline{t} \times \underline{I}) dV$$

$$= \int_V \frac{\partial}{\partial x_k} (\epsilon_{ijk} \underline{t}_j \underline{I}_k) dV$$

$$\Rightarrow \frac{\partial}{\partial t} \left(\varepsilon_{ijk} x_j (p_{uk}) \right) + \frac{\partial}{\partial x_l} \left(\varepsilon_{ijk} x_j u_k (p_{ul}) \right) = \varepsilon_{ijk} x_j f_k$$

$$+ \frac{\partial}{\partial x_l} \left(\varepsilon_{ijk} x_j T_{kl} \right)$$

for $x_j = 0$

— (2)

c) SUBTRACTING (1) FROM (2)

$$- \hat{e}_i \varepsilon_{lji} \left(\frac{\partial}{\partial x_k} (p_{ul} u_k) \right) x_j + \frac{\partial}{\partial x_l} \left(\varepsilon_{ijk} x_j u_k (p_{ul}) \right) \hat{e}_i$$

$$= \hat{e}_i \varepsilon_{ijk} \frac{\partial}{\partial x_l} (x_j T_{kl}) - \varepsilon_{lji} x_j \frac{\partial T_{ik}}{\partial x_k} \hat{e}_l$$

SWITCHING DUMMY INDICES

$$\Rightarrow - \varepsilon_{ijk} \frac{\partial}{\partial x_l} (p_{uk} u_l) x_j + \varepsilon_{ijk} \frac{\partial}{\partial x_l} (p_{ul} u_k x_j)$$

$$= \varepsilon_{ijk} \frac{\partial}{\partial x_l} (x_j T_{kl}) - \varepsilon_{ijk} x_j \frac{\partial T_{kl}}{\partial x_l}$$

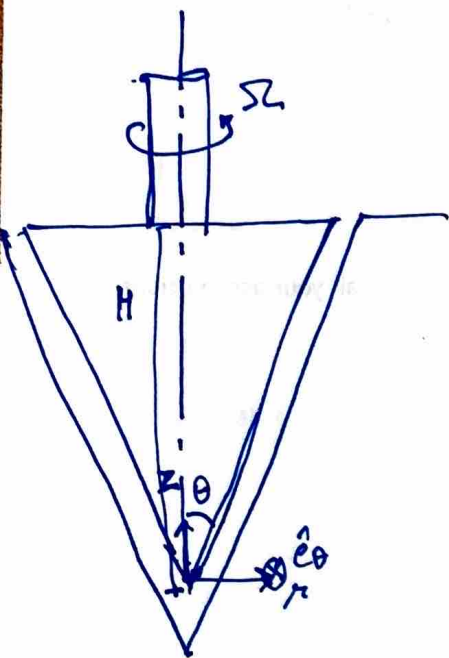
$$p_{ul} u_k \varepsilon_{ijk} \frac{\partial x_j}{\partial x_l} = \varepsilon_{ijk} T_{kl} \frac{\partial x_j}{\partial x_l}$$

$$\varepsilon_{ilk} p_{ulle} = \varepsilon_{ilk} T_{kl}$$

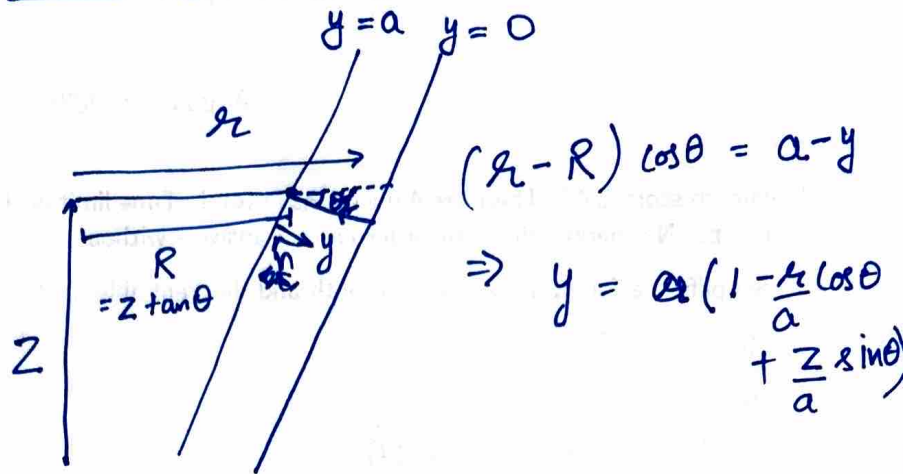
$$\varepsilon_{ilk} p_{ulle} = \varepsilon_{ike} u_{lle} = - \varepsilon_{ike} u_{kle} = 0$$

$$\Rightarrow \varepsilon_{ilk} T_{kl} = 0 \Rightarrow \boxed{T_{kl} = T_{lk}}$$

For problem 4, see Kundu and Cohen's textbook



METHOD-1 (LONG BUT INSTRUCTIVE)



CLEARLY $\underline{V} = V_\theta \hat{e}_\theta$

ASSUME LINEAR VARIATION OF V_θ WITH y

$$V_\theta = Ay + B \quad V_\theta(y=0) = 0 \Rightarrow B = 0$$

$$V_\theta(y=a) = R\Omega \Rightarrow A = \frac{R\Omega}{a}$$

$$\Rightarrow V_\theta = \frac{R\Omega y}{a} = \frac{R\Omega}{a} \left(1 - \frac{r \cos \theta}{a} + \frac{z \sin \theta}{a} \right)$$

a) SHEAR STRESS

ONLY $\sigma_{r\theta} = \mu r \frac{\partial}{\partial r} \left(\frac{V_\theta}{r} \right)$, $\sigma_{\theta z} = \mu \frac{\partial V_\theta}{\partial z}$
ARE NON-ZERO (CONFIRM FOR YOURSELF)

$$\sigma_{r\theta} = \mu r \frac{\partial}{\partial r} \left(\frac{R\Omega}{a} \left(1 - \frac{r \cos \theta}{a} + \frac{z \sin \theta}{a} \right) \right) = \mu r \left(-\frac{\Omega z \tan \theta}{r^2} - \frac{\Omega z^2 \sin^2 \theta}{a r^2 \cos \theta} \right)$$

$$\sigma_{\theta z} = \mu \frac{\partial V_\theta}{\partial z} = \mu \frac{\partial}{\partial z} \left(\Omega z \tan \theta - \frac{\Omega z r \sin \theta}{a} + \frac{\Omega z^2 \sin^2 \theta}{a \cos \theta} \right)$$

$$= \mu \left(\Omega \tan \theta - \frac{\Omega r \sin \theta}{a} + 2 \frac{\Omega z \sin^2 \theta}{a \cos \theta} \right)$$

FOR $r = R = z \tan \theta$ (ON CONE SURFACE)

$$\sigma_{r\theta} = -\mu \Omega \left(1 + \frac{z \sin \theta}{a} \right), \quad \sigma_{\theta z} = \mu \left(\Omega \tan \theta + \frac{\Omega z \sin^2 \theta}{a \cos \theta} \right)$$

b) $\hat{n} = \hat{e}_H \cos \theta - \hat{e}_z \sin \theta$ (VISUALIZE FOR $\theta = 0, \pi/2$)

$$\underline{\underline{t}} = \underline{\underline{\sigma}} \cdot \hat{n} = \sigma_{H0} \cos \theta \hat{e}_\theta + \sigma_{H4} \cos \theta \hat{e}_H + \sigma_{H2} \cos \theta \hat{e}_z - \sigma_{z0} \sin \theta \hat{e}_\theta - \sigma_{zH} \sin \theta \hat{e}_H - \sigma_{z2} \sin \theta \hat{e}_z$$

$$= (\sigma_{H0} \cos \theta - \sigma_{z0} \sin \theta) \hat{e}_\theta$$

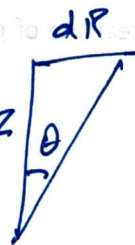
$$= \left[\left(-\mu \Sigma \cos \theta - \mu \Sigma \frac{\sin^2 \theta}{\cos \theta} \right) - \frac{\mu \Sigma z \sin \theta \cos \theta}{a} - \frac{\mu \Sigma z \sin^3 \theta}{a \cos \theta} \right] \hat{e}_\theta$$

$$= \left[-\frac{\mu \Sigma}{\cos \theta} - \frac{\mu \Sigma z \tan \theta}{a} \right] \hat{e}_\theta$$

$$= - \left[\frac{\mu \Sigma}{\cos \theta} + \frac{\mu \Sigma z \tan \theta}{a} \right] \hat{e}_\theta$$

$$M = - \int_{\text{CONE}} |\underline{\underline{t}}| R dS = + \int \left[\frac{\mu \Sigma}{\cos \theta} + \frac{\mu \Sigma z \tan \theta}{a} \right] R dS$$

$$dS = 2\pi R \frac{dR}{\sin \theta} = 2\pi z dz \frac{\tan \theta}{\cos \theta} dz$$



$$M = \int_0^{H \tan \theta} \left[\frac{\mu \Sigma}{\cos \theta} + \frac{\mu \Sigma z \tan \theta}{a} \right] 2\pi R \frac{dR}{\sin \theta}$$

$$= \frac{2\pi \mu \Sigma (H \tan \theta)^3}{\sin \theta} \left[\frac{1}{3 \cos \theta} + \frac{H \tan \theta}{4a} \right]$$

For $a \ll H \tan \theta$

$$M \approx \frac{2\pi \mu \Sigma (H \tan \theta)^4}{4a \sin \theta}$$

METHOD 2 (IF YOU THINK YOU CAN BE A GOOD ENGINEER WITH JEE LEVEL "INTUITION")

$$\tau = \mu \frac{du}{dy} \approx \mu \frac{\Delta u}{\Delta y} = \mu \frac{(R\omega - 0)}{a} = \mu \frac{R\omega}{a}$$

$$\text{MOMENT} = \int \tau R ds = \int \frac{\mu R\omega}{a} 2\pi R \frac{dR}{\sin\theta}$$

$$M = \frac{2\pi\mu\omega}{4a\sin\theta} (H + \tan\theta)^4 = \int_0^{H+\tan\theta} \frac{2\pi\mu R^3\omega}{a\sin\theta} dR$$

WHAT THIS METHOD DOESN'T TEACH YOU

- USAGE OF $\underline{\tau} = \underline{T} \cdot \hat{n}$ (FOR SHEAR TRACTION)
 $\underline{T} = -p\underline{I} + \underline{\sigma}$ $\underline{\tau} = \underline{\sigma} \cdot \hat{n}$ IS SUFFICIENT)

- $\mu \frac{\Delta u}{\Delta y}$ IS SHEAR, BUT SHEAR CAN HAVE MULTIPLE ORIENTATIONS DEPENDING ON $\hat{n}$

