

Tutorial 3 : Dynamical laws and viscometry

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1. Consider the differential form of continuity equation. Show that $\nabla \cdot \underline{v}$ is equal to the bulk strain rate of a fluid element.
2. Through this problem, we will derive the rate of change of any intensive property in an arbitrary moving control volume.

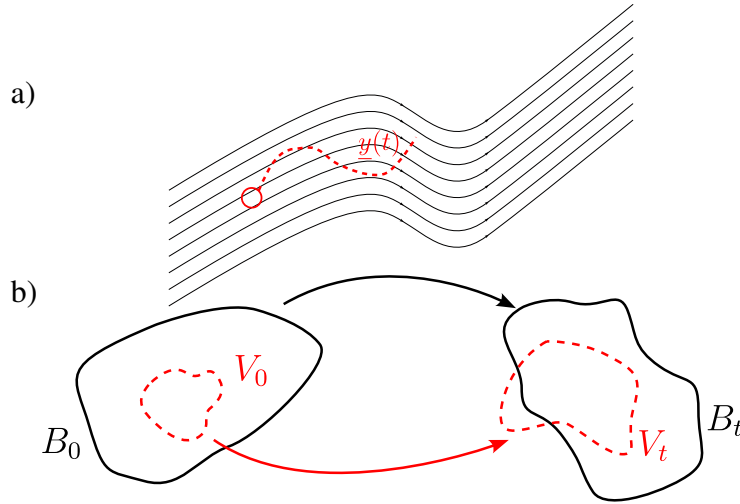


Figure 1: a) An arbitrary probe moving with coordinates $\underline{y}(t)$ in a flow. A fluid blob in initial and deforming configurations (B_0 and B_t respectively) with an arbitrary volume in reference and deforming configurations (V_0 and V_t respectively).

- (a) First, consider figure 1a). A probe which measures some intensive quantity ϕ in a flow moves with coordinate at time t given by $\underline{y}(t)$. The field measured by the probe is $\phi(\underline{y}(t), t)$. Compute the rate of change of ϕ as measured. Take velocity of probe $\dot{\underline{y}}(t) = \underline{w}$.
- (b) Now let us consider a fluid blob at instantaneous configuration B_t and some initial configuration B_0 . Also, there is an arbitrary volume V_t with an initial volume V_0 . The deformation and movement of V_t is our choice and is in no way connected to that of the fluid. The initial configuration of the volume V_0 is only required to compute the final result. Let the coordinates in V_t be denoted by $\underline{y}$. We can write the total extensive quantity of the fluid inside the volume V_t as,

$$\int_{V_t} \rho \phi(\underline{y}, t) dV = \int_{V_0} \rho \phi(\underline{y}, t) J_V dV_0, \quad (1)$$

where J_V is the ratio of volumes dV_t/dV_0 . Using $\frac{dJ_V}{dt}$, show that,

$$\frac{d}{dt} \int_{V_t} \rho dV = - \int_{V_t} \nabla_{\underline{y}} \cdot (\rho(\underline{v} - \underline{w})) dV = - \oint_{S_V} \rho(\underline{v} - \underline{w}) \cdot \hat{n} dS, \quad (2)$$

where $\nabla_{\underline{y}}$ denotes differentiation w.r.t. $\underline{y}$ vector, S_V is the surface of V_t , $\underline{v}$ is the velocity of any fluid particle at $\underline{y}$ at time t , and $\underline{w}$ is the velocity of the point $\underline{y}$ due to movement/deformation of the volume V_t .

3. (See R. L. Panton, Section 5.8) The Newton's second law for control volumes is obtained by considering the intensive property $\phi = v_i$ in the RTT. Decomposing forces in the surface and volume forces, we obtain,

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial}{\partial x_j} (\rho v_i u_j) = f_i + \frac{\partial T_{ij}}{\partial x_j}, \quad (3)$$

as obtained in class. T_{ij} is the surface stress tensor.

- Consider the above equation is written at a point with position vector $\underline{r} = x_i \hat{e}_i$. Take the cross-product of the position vector with the above equation to obtain an equation for the moment of momentum $\epsilon_{ijk} x_j v_k$.
 - Replace ϕ with $\epsilon_{ijk} x_j v_k + \alpha_i$ in the RTT to obtain an equation for the angular momentum.
 - Show that for non-polar fluids (no moments on the molecules, no rotational acceleration $\alpha_i = 0$), $T_{ij} = T_{ji}$.
4. (See Kundu and Cohen, *Fluid Mechanics*, 2nd edition, Section 13 and 14, Chapter 4) Kinetic energy of a system is defined as $\Phi = \frac{1}{2} m |\underline{v}|_{cm}^2$. The corresponding intensive quantity (kinetic energy per unit mass) is $\phi = \frac{1}{2} |\underline{v}|^2$ for velocity field $\underline{v}$. The kinetic energy per unit volume is $\rho\phi$. Starting from Equation 3,
- Derive the governing equation for kinetic energy per unit mass by taking a dot product of Equation 3 with $\underline{v}$.
 - Using the continuity equation, derive the governing equation for kinetic energy per unit volume.
 - Provide a physical interpretation of the term $\underline{v} \cdot (\nabla \cdot \underline{T})$. Consider a box-shaped fluid element. Using specific components of $\underline{v}$ and $\underline{T}$, show that the term $\underline{v} \cdot (\nabla \cdot \underline{T})$ denotes the rate of mechanical work done.
 - The rate form of first law of thermodynamics is given by,

$$\frac{dE_T}{dt} = \frac{\delta Q}{\delta t} + \frac{\delta W}{\delta t}, \quad (4)$$

where δ has been used instead of d to denote infinitesimal change in path-dependent quantities. Quantity E_T is the extensive total energy given by

$$E_T = E + \frac{1}{2} m |\underline{v}|_{cm}^2, \quad (5)$$

where E is the internal energy. The corresponding intensive quantity can be written as,

$$e + \frac{1}{2} |\underline{v}|^2, \quad (6)$$

for $\underline{v}$ velocity field. Simplify the rate of work done term $(\delta W/\delta t)$ by considering forces on a control volume and using velocity at that location.

- Let the heat addition term $\delta Q/\delta t$ be written as

$$\frac{\delta Q}{\delta t} = - \oint_S \underline{q} \cdot \hat{n} dS. \quad (7)$$

Write the simplified differential form of the total energy equation (both per unit mass and per unit volume).

- Subtract the kinetic energy (per unit volume) equation from the total energy equation (per unit volume) to derive the equation for internal energy (both per unit mass and per unit volume).

(g) Show that the remaining term from the rate of work done contribution in the internal energy is

$$T_{ij} \frac{\partial v_i}{\partial x_j} = \underline{\underline{T}} : \nabla \underline{\underline{v}}. \quad (8)$$

Discuss its physical significance in a similar way as done above using a box-shaped fluid element.

(h) The rate form of the combined second law and first law of thermodynamics is given by,

$$T \frac{Ds}{Dt} = \frac{De}{Dt} - \frac{p}{\rho^2} \frac{D\rho}{Dt}, \quad (9)$$

where s is specific entropy. Using the continuity and the internal energy equation, show that the rate of increase of specific entropy for a fluid element is always positive in the absence of heat addition $\delta Q/\delta t$.

5. A conical pointed shaft turns in a conical bearing. The gap between the shaft and the bearing is filled with heavy oil of viscosity $\mu = 0.2 \text{ N}\cdot\text{s}/\text{m}^2$.

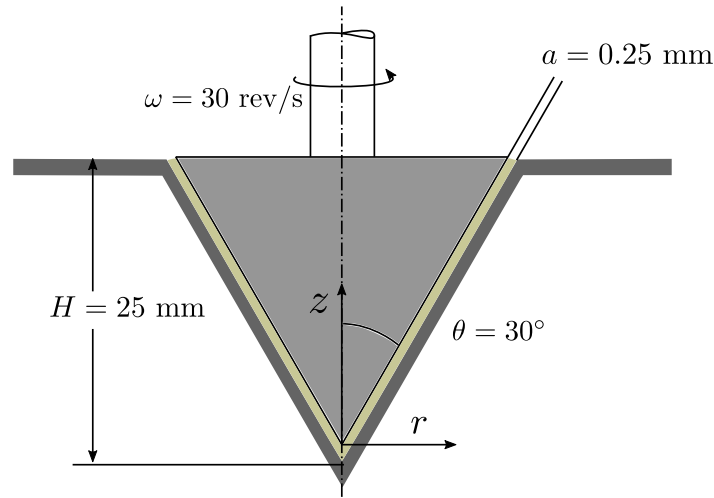


Figure 2: Conical shaft in a conical bearing.

- Obtain an algebraic expression for the shear stress that acts on the surface of the conical shaft.
 - Calculate the viscous torque required to keep the shaft rotating.
6. A cylindrical shaft turns in a bearing. The gap between the shaft and the bearing is filled with an oil of viscosity μ . Calculate the moment M required to keep the shaft rotating at an angular speed of Ω .

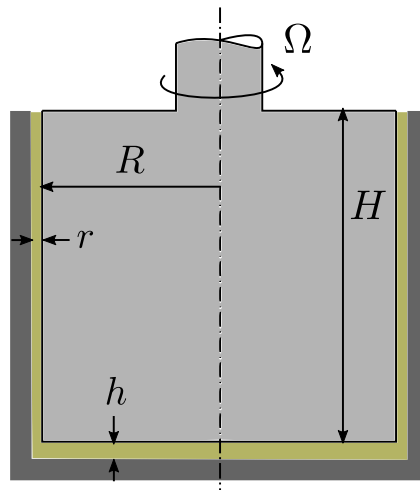


Figure 3: Cylindrical shaft in a cylindrical bearing

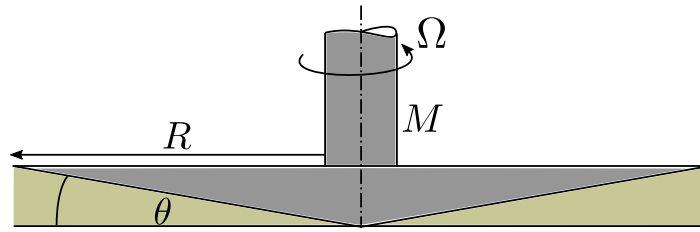


Figure 4: Cone-plate viscometer.

7. The device show in figure 4 is called a cone-plate viscometer. The angle of the cone is very small and the gap is filled with the test liquid. The torque M is required to rotate the cone at a rate Ω is measured. Assuming a linear viscosity profile in the fluid film, derive an expression for the fluid viscosity μ as a function of M, R, Ω, θ .