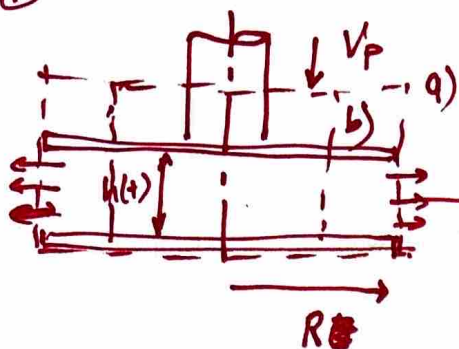


①



a) $\frac{d}{dt} \int_V \rho dV = - \int_S \rho (\underline{v} - \underline{w}) \cdot \hat{n} dS$

$\frac{d}{dt} \rho (\pi R^2 h) = - \rho V_h 2\pi R h$

$\frac{dh}{dt} = -V_p$

$\Rightarrow \rho \pi R^2 V_p = + \rho V_h 2\pi R h$

$h = h_0 - V_p t$

$\Rightarrow V_h = \frac{R V_p}{2h} = \frac{R V_p}{2(h_0 - V_p t)}$

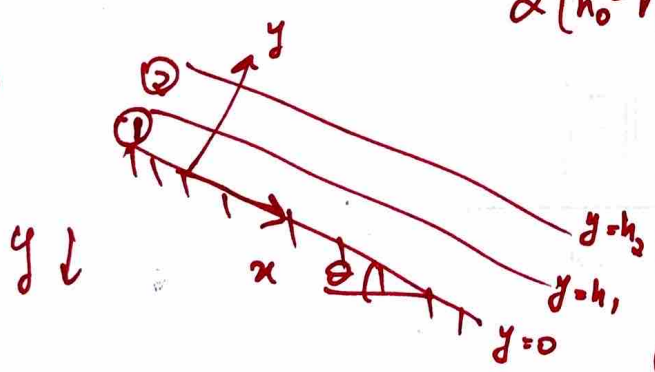
b) UNIFORM IN NORMAL $\rightarrow$ (WALL NORMAL)

$\rightarrow$ INVISCID

CONSIDER CV OF SOME RADIUS R

$\Rightarrow V_h = \frac{R V_p}{2(h_0 - V_p t)} = u(h, t)$

②



ASSUMPTIONS

① STEADY FLOW $\Rightarrow \frac{\partial}{\partial t} = 0$

② FULLY DEVELOPED $\Rightarrow \frac{\partial \underline{v}}{\partial x} = 0$

③ 2D Flow $\Rightarrow \underline{v} = u\hat{i} + v\hat{j}$

CONTINUITY : $\nabla \cdot \underline{v} = 0 \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$\Rightarrow v = \text{const in } ① \text{ \& } ②$

AT $y=0$ $v_① = 0$
 $\Rightarrow v_② = 0$

AT $y=h$, $v_① = v_② \Rightarrow v_② = 0 \Rightarrow$ BOTH STREAMS ARE UNIDIRECTIONAL

$$y: \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \rho g \cos \theta$$

IN BOTH STREAMS $\frac{\partial p_1}{\partial y} = \rho_1 g \cos \theta$

$$\frac{\partial p_2}{\partial y} = \rho_2 g \cos \theta$$

AT $y = h_2$ $p_{(2)} = p_{atm} = \text{const.}$

$$\Rightarrow p_{(2)} = p_{atm} + \rho_2 g \cos \theta y + f(x)$$

$f(x) = 0$ SINCE $p_{(2)} = p_{atm} = \text{const}$ AT $y = h_2$

$$p_{(1)} = p_{(2)}(y=h_1) + \rho_1 g \cos \theta y$$

$$p_{(1)}(y=h_1) = p_{(2)}(y=h_1) = p_{atm} + \rho_2 g \cos \theta h_1$$

$$p_{(1)} = p_{atm} + \rho_2 g \cos \theta h_1 + \rho_1 g \cos \theta y$$

IN BOTH STREAMS $\frac{\partial p_1}{\partial x} = 0, \frac{\partial p_2}{\partial x} = 0$

$$x: \rho \left(\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial y} + u \frac{\partial u}{\partial x} \right) = - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \rho g \sin \theta$$

$$\Rightarrow \mu_1 \frac{d^2 u_{(1)}}{dy^2} = - \rho_1 g \sin \theta, \quad \mu_2 \frac{d^2 u_{(2)}}{dy^2} = - \rho_2 g \sin \theta$$

$$\Rightarrow u_{(1)} = - \left(\frac{\rho_1 g \sin \theta}{\mu_1} \right) \frac{y^2}{2} + A_1 y + B_1$$

4 CONDITIONS
REQUIRED

$$u_{(2)} = - \left(\frac{\rho_2 g \sin \theta}{\mu_2} \right) \frac{y^2}{2} + A_2 y + B_2$$

$$u_{(1)}(y=0) = 0 \quad (\text{NO SLIP})$$

LET $\frac{\rho_1 g \sin \theta}{\mu_1} = \lambda_1$

$\frac{\rho_2 g \sin \theta}{\mu_2} = \lambda_2$

NO SLIP AT INTERFACE $u_{(1)}(y=h_1) = u_{(2)}(y=h_1)$

SHEAR STRESS AT INTERFACE $\mu_1 \frac{du_{(1)}}{dy}(y=h_1) = \mu_2 \frac{du_{(2)}}{dy}(y=h_1)$

FREE SHEAR AT $y=h_2$ $\frac{du_{(2)}}{dy}(y=h_2) = 0$

$$u_0(y=0) = 0 \Rightarrow \boxed{B_1 = 0}$$

$$\frac{du_2}{dy}(y=h_2) \approx 0 \Rightarrow \boxed{A_2 = \lambda_2 h_2}$$

$$u_0(y=h_1) = u_2(y=h_1)$$

$$\Rightarrow -\lambda_1 \frac{h_1^2}{2} + A_1 h_1 = -\lambda_2 \frac{h_1^2}{2} + \lambda_2 h_1 h_2 + B_2$$

$$A_1 h_1 - B_2 = \left(\frac{\lambda_1 h_1^2}{2} - \frac{\lambda_2 h_1^2}{2} \right) + \lambda_2 h_1 h_2$$

$$\mu_1 \frac{du_0}{dy} \Big|_{(y=h_1)} = \mu_2 \frac{du_2}{dy} \Big|_{(y=h_1)}$$

$$\Rightarrow \mu_1 (-\lambda_1 h_1 + A_1) = \mu_2 (-\lambda_2 h_1 + A_2) \quad \underbrace{}_{\rightarrow \lambda_2 h_2}$$

$$\Rightarrow A_1 - \lambda_1 h_1 = \frac{\mu_2}{\mu_1} (\lambda_2 h_2 - \lambda_2 h_1)$$

$$A_1 = \lambda_1 h_1 + \frac{\mu_2}{\mu_1} (\lambda_2) (h_2 - h_1)$$

$$B_2 = A_1 h_1 - \lambda_2 h_1 h_2 + \frac{\lambda_2}{2} (\lambda_2 - \lambda_1) \frac{h_1^2}{2}$$

$$= \lambda_1 h_1^2 + \frac{\mu_2 \lambda_2}{\mu_1} (h_2 h_1 - h_1^2) - \lambda_2 h_1 h_2 + (\lambda_2 - \lambda_1) \frac{h_1^2}{2}$$

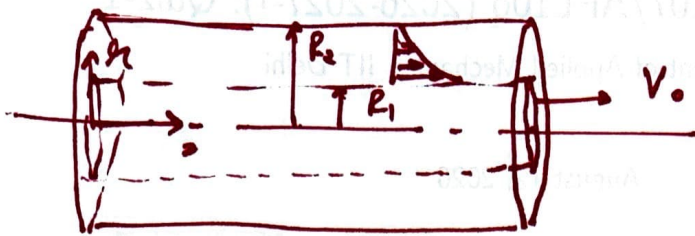
$$= (\lambda_2 + \lambda_1) \frac{h_1^2}{2} + \lambda_2 h_2 h_1 \left(\frac{\mu_2}{\mu_1} - 1 \right) - \frac{\mu_2 \lambda_2}{\mu_1} h_1^2$$

~~SUBSTIT~~ NO NEED TO SUBSTITUTE A_1, A_2, B_1, B_2 BACK

IN EXAMS IN SUCH

LENGTHY CALCULATIONS

⑤



ASSUMPTIONS

① STEADY $\Rightarrow \frac{\partial(\cdot)}{\partial t} = 0$

② FULLY DEVELOPED

IN $z \Rightarrow \frac{\partial(V)}{\partial z} = 0$

③ 2D FLOW

$\underline{V} = V_r \hat{e}_r + V_z \hat{e}_z$

④ AXISYMMETRIC

$\frac{\partial(\cdot)}{\partial \theta} = 0$

CONTINUITY

$$\frac{1}{r} \frac{\partial}{\partial r} (r V_r) + \frac{\partial V_z}{\partial z} = 0$$

$\Rightarrow r V_r = \text{const.}$

AT $r = R_1$ $V_r = 0$ (SOLID WALL)

$\Rightarrow V_r = 0$

$\Rightarrow$ UNI DIRECTIONAL FLOW

r : $\rho \left(\frac{\partial V_r}{\partial t} + \frac{V_r}{r} \frac{\partial V_r}{\partial r} + V_r \frac{\partial V_r}{\partial r} + \frac{V_z}{r} \frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r} \right) = -\frac{\partial p}{\partial r} + \mu \left(\nabla^2 V_r \right)$

$\Rightarrow \frac{\partial p}{\partial r} = 0$

$\Rightarrow p = p(z)$

z : $\rho \left(\frac{\partial V_z}{\partial t} + \frac{V_r}{r} \frac{\partial V_z}{\partial r} + V_z \frac{\partial V_z}{\partial z} + \frac{V_\theta}{r} \frac{\partial V_z}{\partial \theta} \right) = -\frac{\partial p}{\partial z} + \mu \left(\nabla^2 V_z \right)$

$\frac{\mu}{r} \frac{d}{dr} \left(r \frac{dV_z}{dr} \right) = \frac{dp}{dz} = \text{const.}$

$r \frac{dV_z}{dr} = \frac{r^2}{2\mu} \frac{dp}{dz} + A \Rightarrow \boxed{V_z = \frac{r^2}{4\mu} \frac{dp}{dz} + A \ln r + B}$

ASSUME FLOW DRIVEN PURELY BY INNER CYLINDER EXTRUSION

i.e. $\frac{dp}{dz} = 0$. YOU CAN FIND A & B USING B.C.s

EVEN WITHOUT THIS ASSUMPTION. BOTH ARE VALID

SOLUTIONS FOR EXAMS.

I KNOW THIS!
YOU HAVE TO
CONFIRM & SHOW
IN EXAM
THAT THIS
IS 0

$$V_2 = A \ln R + B \quad \left(\text{ASSUME } \frac{dP}{dz} = 0 \right)$$

$$V_2(R=R_1) = V_0 = A \ln R_1 + B$$

$$A \ln \left(\frac{R_1}{R_2} \right) = V_0$$

$$V_2(R=R_2) = 0 = A \ln R_2 + B$$

$$A = \frac{V_0}{\ln \left(\frac{R_1}{R_2} \right)}$$

$$B = -A \ln R_2 = -V_0 \frac{\ln R_2}{\ln \left(\frac{R_1}{R_2} \right)}$$

$$V_2 = V_0 \frac{\ln R}{\ln \left(\frac{R_1}{R_2} \right)} - V_0 \frac{\ln R_2}{\ln \left(\frac{R_1}{R_2} \right)} = V_0 \frac{\ln \left(R/R_2 \right)}{\ln \left(R_1/R_2 \right)}$$

ALTHOUGH IT IS NOT ASKED, BUT YOU CAN ALSO SEE THE ANSWER FOR A

SPECIAL CASE OF SMALL GAP i.e. $\frac{R_2 - R_1}{R_1} \ll 1$

$$\text{LET } R_2 = (1+\epsilon) R_1 \quad \text{FOR } \epsilon \ll 1 \text{ (FIXED)}$$

$$\text{LET } \xi = \frac{R - R_1}{R_1} \Rightarrow 0 < \xi < \epsilon$$

(ANOTHER
COORDINATE.
NOT POTENTIAL!)

$$\Rightarrow R = R_1 (1 + \xi)$$

$$\ln \frac{R}{R_2} = \ln \left[\frac{R_1}{R_2} (1 + \xi) \right] = \ln \frac{R_1}{R_2} + \ln(1 + \xi)$$

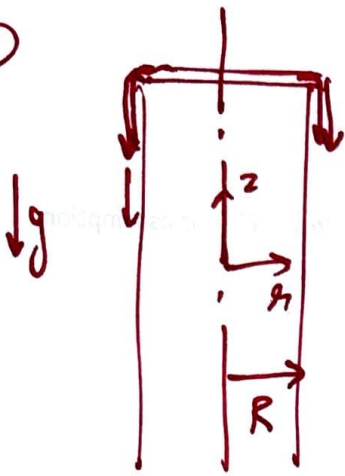
$$\text{SINCE } \xi \ll 1 \Rightarrow \ln(1 + \xi) = \xi + O(\xi^2)$$

$$\ln \left(\frac{R_1}{R_2} \right) = \ln \left(\frac{1}{1 + \epsilon} \right) = -\ln(1 + \epsilon) = -\epsilon + O(\epsilon^2)$$

$$\Rightarrow V_2 = V_0 \left(\frac{\xi - \epsilon}{-\epsilon} \right) = V_0 \left(1 - \frac{\xi}{\epsilon} \right) \rightarrow \text{LINEAR}$$

$$\text{IF GAP } \epsilon R_1 = d, \quad \xi = \frac{R}{R_1} - 1 \Rightarrow V_2 = \left(1 - \left(\frac{R}{R_1} - 1 \right) \right)$$

⑦



ASSUMPTIONS

① STEADY: $\frac{\partial}{\partial t} = 0$

② 2D FLOW $\underline{V} = V_r \hat{e}_r + V_z \hat{e}_z$

③ FULLY DEVELOPED IN z $\frac{\partial V}{\partial z} = 0$

④ AXI SYMMETRIC

CONTINUITY

$$\nabla \cdot \underline{V} = 0$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r V_r) + \frac{\partial V_z}{\partial z} = 0$$

$$r V_r = \text{const.}$$

$$V_r = 0 \text{ AT } r = R$$

$$\Rightarrow V_r = 0 \text{ EVERYWHERE}$$

USE A MOMENTUM EQUATION

IN EXAM TO SHOW $\frac{\partial p}{\partial r} = 0$

$$z: \rho \left(\frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} + V_z \frac{\partial V_z}{\partial z} + \frac{V_\theta}{r} \frac{\partial V_z}{\partial \theta} \right) = - \frac{\partial p}{\partial z} + \mu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V_z}{\partial \theta^2} \right) - \rho g$$

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dV_z}{dr} \right) = \frac{dp}{dz} + \rho g$$

SAME AS PROBLEM ⑤
COMMENT

$$\text{LET } \frac{dp}{dz} = 0$$

$$r \frac{dV_z}{dr} = \frac{\rho g}{4\mu} r^2 + A \Rightarrow V_z = \frac{\rho g}{4\mu} r^2 + A \ln r + B$$

$$V_z(r=R) = 0 \Rightarrow \frac{\rho g R^2}{4\mu} + A \ln R + B = 0$$

SHEAR AT $r = R + \delta$

$$\sigma_{rz} = \mu \left(\frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r} \right) \approx \mu_{\text{air}} \frac{\partial V_z}{\partial r} \Big|_{\text{air}} \Rightarrow \frac{\partial V_z}{\partial r} \approx 0 \text{ AT } R + \delta$$

$$\frac{\rho g}{2\mu} (R + \delta) + \frac{A}{(R + \delta)} = 0 \Rightarrow A = - \frac{\rho g}{2\mu} (R + \delta)^2 \Rightarrow B = - \frac{\rho g R^2}{2\mu} \left(\frac{1}{2} - \left(\frac{R + \delta}{R} \right)^2 \right)$$

$$V_z = \frac{\rho g}{2\mu} \left(\frac{r^2}{2} - (R+\delta)^2 \ln r \right) - \frac{\rho g}{2\mu} \left(\frac{R^2}{2} - \left(1 + \frac{\delta}{R}\right)^2 \ln R \right)$$

$$= \frac{\rho g}{4\mu} (r^2 - R^2) + \frac{\rho g R^2}{2\mu} \left(\left(1 + \frac{\delta}{R}\right)^2 \ln R - \left(1 + \frac{\delta}{R}\right)^2 \ln r \right)$$

$$= \frac{\rho g}{4\mu} (r^2 - R^2) + \frac{\rho g R^2}{2\mu} \left(1 + \frac{\delta}{R}\right)^2 \ln \frac{R}{r}$$

$$\int_R^{R+\delta} V_z 2\pi r dr = Q \rightarrow \text{FIND } \delta$$

(COMPLETE ALGEBRA IN
EXAM FOR FULL
MARKS)