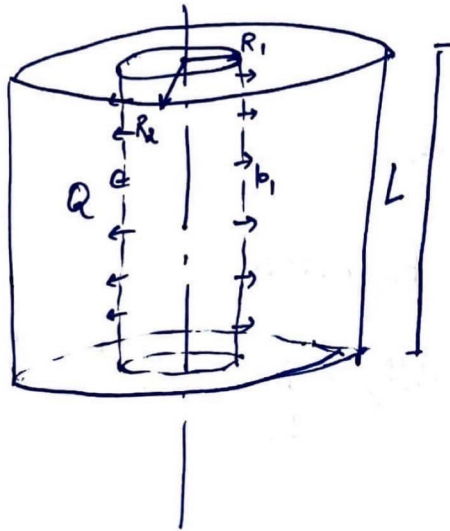


4



ASSUMING AXISYMMETRIC FLOW

$$\frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial u_r}{\partial r}) + \frac{\partial u_z}{\partial z} = 0$$

ID (RADIAL) FLOW.

$$\Rightarrow \frac{1}{r} \frac{\partial}{\partial r} (r u_r) = 0 \Rightarrow u_r = \frac{C}{r}$$

$$\oint u_r (2\pi R_1) L = Q \Rightarrow \boxed{C = \frac{Q}{2\pi L}}$$

MOMENTUM (STEADY)

$$u_r \frac{\partial u_r}{\partial r} + \cancel{\frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta}} + u_z \frac{\partial u_r}{\partial z} - \cancel{\frac{u_\theta^2}{r}} = -\frac{1}{\rho} \frac{dp}{dr} + \nu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_r}{\partial r} \right) - \frac{u_r^2}{r^2} \right)$$

$$u_r \frac{\partial u_r}{\partial r} + \frac{1}{\rho} \frac{dp}{dr} = \frac{\nu}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_r}{\partial r} \right) - \frac{u_r^2}{r^2}$$

$$u_r = \frac{C}{r}, \quad \frac{\partial u_r}{\partial r} = -\frac{C}{r^2}$$

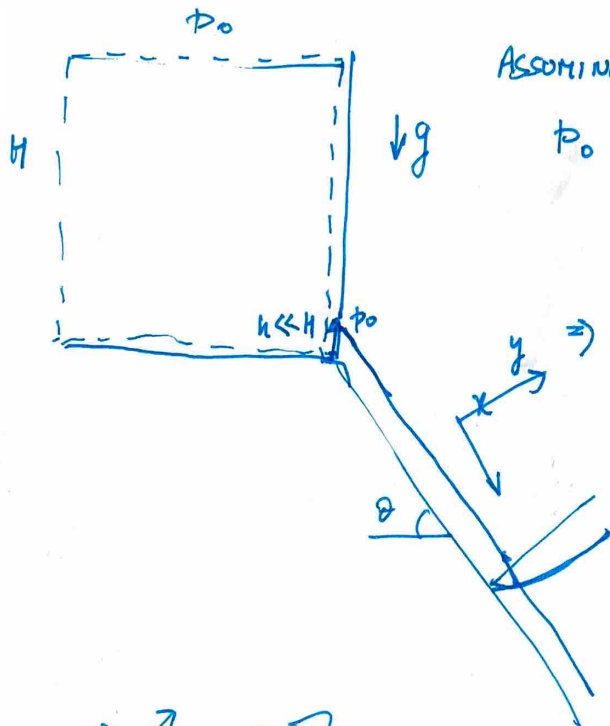
$$-\frac{C^2}{r^3} + \frac{1}{\rho} \frac{dp}{dr} = \frac{\nu}{r} \frac{d}{dr} \left(-\frac{C}{r} \right) - \frac{\nu C}{r^3}$$

$$\frac{1}{\rho} \frac{dp}{dr} = \frac{\nu C}{r^3} - \frac{\nu C}{r^3} + \frac{C^2}{r^3} \Rightarrow \frac{dp}{dr} = \frac{\rho C^2}{r^3}$$

$$\Rightarrow \int_{p_1}^p dp = \rho C^2 \int_{R_1}^r \frac{dr}{r^3} \Rightarrow p - p_1 = \rho C^2 \left(\frac{1}{R_1^2} - \frac{1}{r^2} \right)$$

$$p = p_1 + \rho \left(\frac{Q}{2\pi L} \right)^2 \left(\frac{1}{R_1^2} - \frac{1}{r^2} \right)$$

3



ASSUMING INVISCID FLOW IN TANK

$$p_0 + \frac{1}{2} \rho v^2 = p_0 + \rho g H$$

$$\Rightarrow v = \sqrt{2gH}$$

$$\Rightarrow \text{FLOW RATE (PER UNIT DEPTH)} = h\sqrt{2gH}$$

SIMPLIFYING N.S. EQUATIONS

- FULLY DEVELOPED
- STEADY

$$x: \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \rho g \sin \theta$$

$$y: \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \rho g \cos \theta$$

$$\Rightarrow \frac{\partial p}{\partial y} = \rho g \cos \theta, \quad \frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} + \rho g \sin \theta$$

$$\text{AT } y=0 \quad (u=0)$$

$$\text{AT } y=d \quad \left(\mu \frac{\partial u}{\partial y} \Big|_{\text{water}} = \mu \frac{\partial u}{\partial y} \Big|_{\text{air}} \right) \Rightarrow \frac{\partial u}{\partial y} \approx 0$$

$p_{\text{water}} = p_{\text{air}}$

ATMOSPHERIC

Since pressure is uniform $p_{air} = p_0 = \text{const.}$

$$\Rightarrow p_{water} = p_0 \text{ AT } y = \delta \quad \forall x \Rightarrow \left. \frac{\partial p}{\partial x} \right|_{y=\delta} = 0$$

$$\frac{\partial p}{\partial y} = -\rho g \cos \theta \Rightarrow p = -\rho g \cos \theta y + C(x)$$

$$p(\delta) = p_0 = C - \rho g \cos \theta \delta.$$

$$\Rightarrow p = p_0 + \rho g \cos \theta (\delta - y) \Rightarrow C(x) = p_0 + \rho g \cos \theta \delta = \text{const.}$$

$$\Rightarrow \frac{\partial p}{\partial x} = 0 \Rightarrow \mu \frac{\partial^2 u}{\partial y^2} = -\rho g \sin \theta$$

$$\Rightarrow u = -\frac{\rho g \sin \theta}{2\mu} y^2 + C_1 y + C_2$$

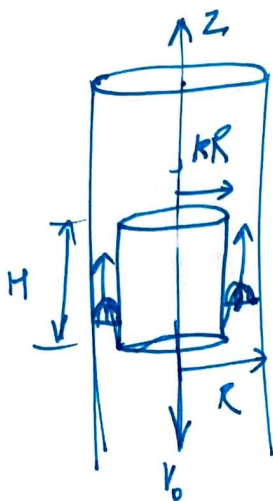
$$u(y=0) = 0 \Rightarrow C_2 = 0, \quad \frac{\partial u}{\partial y} = -\frac{\rho g \sin \theta}{\mu} y + C_1$$

$$\left. \frac{\partial u}{\partial y} \right|_{\delta} = 0 \Rightarrow C_1 = \frac{\rho g \sin \theta}{\mu} \delta \Rightarrow u = \frac{\rho g \sin \theta}{\mu} \left(\delta y - \frac{y^2}{2} \right)$$

$$\int_0^{\delta} u dy = \frac{\rho g \sin \theta}{\mu} \left(\frac{\delta^3}{2} - \frac{\delta^3}{6} \right) = \frac{\rho g \sin \theta \delta^3}{3\mu} = h \sqrt{2gH}$$

(FIND δ)

6



IN CYLINDRICAL COORDINATES

• ASSUME AXISYMMETRIC FLOW $\frac{\partial}{\partial \theta} = 0$ • $u_\theta = 0$

$$\frac{1}{r} \frac{\partial}{\partial r} (r u_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

• FULLY DEVELOPED OVER

$$\Rightarrow r u_r = \text{const.}$$

SLUG HEIGHT $\Rightarrow \frac{\partial}{\partial z} = 0$

$$u_r (r = kR) = 0$$

• STEADY

$$\Rightarrow u_r = 0$$

$$\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} \right)$$

$$\Rightarrow \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) = \frac{1}{\mu} \frac{\partial p}{\partial z}$$

FROM θ & r : MOMENTUM EQUATIONS

$$\frac{\partial p}{\partial r} = 0, \quad \frac{\partial p}{\partial \theta} = 0$$

$$\Rightarrow \frac{\partial p}{\partial z} = \frac{dp}{dz} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) = \text{const.}$$

(SHOW THIS IN EXAMS!)

$$\Rightarrow \frac{dp}{dz} \frac{r^2}{2} + C_1 = r \frac{\partial u_z}{\partial r} \Rightarrow u_z = \frac{dp}{dz} \frac{r^2}{4} + C_1 \ln r + C_2$$

BOUNDARY CONDITIONS

$$u_z (r = kR) = -V_0, \quad u_z (r = R) = 0$$

$$\frac{dp}{dz} (kR)^2 + C_1 \ln(kR) + C_2 = -V_0 \Rightarrow V_0 = C_1 (\ln R - \ln kR) + \frac{dp}{dz} (R^2 - k^2 R^2)$$

$$\frac{dp}{dz} R^2 + C_1 \ln R + C_2 = 0$$

$$\Rightarrow C_1 = \frac{V_0}{\ln R - \ln kR} - \frac{dp}{dz} \frac{R^2 (1 - k^2)}{\ln R - \ln kR}$$

$$C_2 = -\frac{dp}{dz} R^2 - C_1 \ln R = -\frac{dp}{dz} R^2 + \frac{dp}{dz} \frac{R^2 (1 - k^2)}{\ln \frac{1}{k}} \ln R - \frac{V_0 \ln R}{\ln \frac{1}{k}}$$

$$\Rightarrow C_2 = \frac{dp}{dz} \frac{R^2}{\ln \frac{1}{k}} \left(\frac{(1 - k^2)}{\ln R} - 1 \right) + \frac{V_0 \ln(R)}{\ln(k)}$$

BUT $\frac{dp}{dz}$ IS NOT KNOWN. $\frac{dp}{dz}$ CAN BE OBTAINED USING MASS

CONSERVATION.

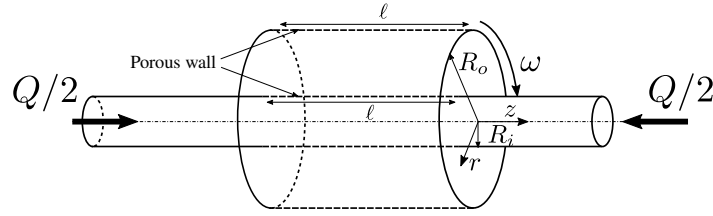


Figure 8: Rotating filter

8. A centrifuge maybe used to filter sediments from a water supply. Consider the setup shown in figure 8 where the water is supplied at flow rate Q from both ends to a pipe which has a permeable wall in a particular section. The outer cylinder, also made of permeable wall is rotating at an angular velocity ω . Clear water transpires through both the pipe and cylinder getting filtered.. The transpiration velocity is measured to be V from the outside cylinder. Assume axisymmetric and flow and evaluate $u_\theta(r)$ and $u_r(r)$ between the cylinders. Follow below steps,

- (a) Apply continuity equation and find u_r . What is the transpiration velocity through the inner cylinder in terms of V ? Can you find V in terms of Q assuming steady state.

Answer:- $u_\theta(R_i) = 0$, $u_\theta(R_o) = R_o\omega$ and $u_r(R_i) = \frac{Q}{2\pi R_i \ell}$. Alternatively $u_r(R_o)$ may be found using continuity but differential equation governing u_r is first order so only 1 boundary condition is required. Assuming axisymmetric: $\frac{\partial \vec{v}}{\partial \theta} = 0$, 2D: $u_z = 0$, $\frac{\partial}{\partial z} = 0$, incompressible : $\nabla \cdot \vec{v} = 0$, steady : $\frac{\partial}{\partial t} = 0$

$$\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{r} \frac{\partial}{\partial r} (r u_r) = 0 \implies r u_r = K = \text{const.} \quad (0.0.1)$$

At $r = R_i$,

$$R_i u_r(R_i) = \frac{Q}{2\pi \ell} \implies K = \frac{Q}{2\pi \ell}. \quad (0.0.2)$$

Hence the transpiration/radial velocity is,

$$u_r = \frac{Q}{2\pi r \ell} \quad (0.0.3)$$

At R_i , $u_r = \frac{Q}{2\pi R_i \ell}$ and R_o , $u_r = \frac{Q}{2\pi R_o \ell} = V$.

- (b) Apply θ momentum equation. Evaluate $u_\theta(r)$. (Hint: If you obtain an ODE which is seemingly unsolvable, substitute $u_\theta = C r^n$ and solve)

Answer:- Substituting the assumptions, r momentum equation simplifies to an ODE for pressure since $\frac{\partial p}{\partial \theta} = 0$ is given.

$$u_r \frac{du_r}{dr} - \frac{v_\theta^2}{r} = -\frac{1}{\rho} \frac{dp}{dr}. \quad (0.0.4)$$

Similarly, from θ momentum equation,

$$u_r \frac{du_\theta}{dr} + \frac{u_r u_\theta}{r} = \nu \left(\frac{d^2 u_\theta}{dr^2} + \frac{1}{r} \frac{du_\theta}{dr} - \frac{u_\theta}{r^2} \right), \quad (0.0.5)$$

we obtain the ODE for u_θ . This may be further simplified to,

$$r^2 \frac{d^2 u_\theta}{dr^2} + r \left(1 - \frac{K}{\nu} \right) \frac{du_\theta}{dr} - \left(1 + \frac{K}{\nu} \right) u_\theta = 0. \quad (0.0.6)$$

Substituting $u_\theta = C r^n$, and $\frac{K}{\nu} = \frac{Q}{2\ell \nu \pi} = \frac{\lambda}{2}$

$$n(n-1) + \left(1 - \frac{\lambda}{2} \right) n - \left(1 + \frac{\lambda}{2} \right) = 0. \quad (0.0.7)$$

The above quadratic simplifies to,

$$n^2 - \frac{\lambda}{2} n - \left(\frac{\lambda}{2} + 1 \right) = 0. \quad (0.0.8)$$

The solutions of n are,

$$n = \frac{\frac{\lambda}{2} \pm \sqrt{\frac{\lambda^2}{4} + 4 - 2\lambda}}{2} = \frac{\lambda}{2} + 1, -1 \quad (0.0.9)$$

Hence the solution of u_θ is,

$$u_\theta = \frac{C_1}{r} + C_2 r^{\frac{\lambda}{2}+1}. \quad (0.0.10)$$

Substituting the boundary conditions,

$$C_2 = \frac{\Omega R_o^2}{R_o^{2+\frac{\lambda}{2}} - R_i^{2+\lambda/2}}, \quad (0.0.11)$$

and

$$C_1 = -\frac{\Omega R_o^2 R^{2+\frac{\lambda}{2}}_i}{R_o^{2+\frac{\lambda}{2}} - R_i^{2+\lambda/2}} \quad (0.0.12)$$

- (c) Apply r momentum equation and evaluate the pressure distribution $p(r)$ between the two cylinders. How does the pressure change correlate with the direction of the flow? **Answer:-** Substitute in r equation to solve for $p(r)$.

$$p = C - \rho \left(\frac{C_1^2}{r} + \frac{K^2}{2r^2} \right) + \rho C_2^2 \frac{r^{3+\lambda}}{3+\lambda} + 2\rho C_1 C_2 r^{1+\frac{\lambda}{2}}. \quad (0.0.13)$$

Find C using pressure at R_i .