

Tutorial 6 : Integral analysis

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1. Water of density ρ_w enters a tank of cross-section area S from the bottom through an inlet of cross-section area A_1 at speed v_1 , initially filled with air of density ρ_A . To let the tank fill, a vent of cross-section area A_2 is provided at the top.

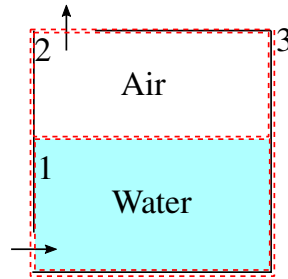


Figure 1: Water filling a tank

- (a) Consider the control volume 1 (enclosing the water contained inside the tank). Apply the mass balance equation. Consider that the instantaneous height of the water is $h(t)$.
 - (b) Consider the control volume 2 (enclosing the air left inside the tank). Apply the mass balance equation to it. Assume that air is leaving at speed v_2 .
 - (c) Derive a relation between v_1 and v_2 .
 - (d) Compute the rate of change of mass of the tank using the above relation.
2. In the Reynolds Transport Theorem, substitute $\beta = v$ where v is the specific volume of the fluid. The obtained equation is called the *Volume continuity* equation. Use this equation to verify the relation obtained in problem 1(c).
 3. Fluid flows between two discs of radius R through an orifice of radius a . If the gap between the discs is also a , find the radial velocity $u(r)$ between the discs.

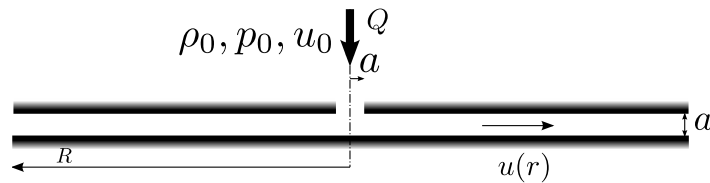


Figure 2: Radial flow between discs

4. Two fluid streams 1 and 2 are mixing in a duct as shown below. Assume that the speed of stream 1 is v_1 , density ρ_1 , and area A_1 (similarly ρ_2 , v_2 , and A_2 for stream 2).

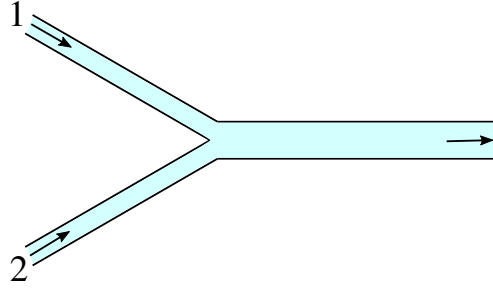


Figure 3: Mixing of two fluid streams

- (a) Draw a suitable control volume. Clearly mark the surfaces of the control volume across which fluid flow occurs.
 - (b) Write the integral form of mass conservation (continuity) equation for this control volume.
 - (c) Solve for the density of the outlet flow, if velocity and area are known.
 - (d) Consider a flow of a mixture of two fluids. Density of the two fluids are ρ_1 and ρ_2 . Derive a suitable set of continuity equation(s) for the flow of a mixture of fluids.
5. For an incompressible flow of a fluid (density ρ) in a pipe of radius R , the velocity profile is given as,

$$u(r) = u_0 \left(1 - \frac{r^2}{R^2} \right).$$

Calculate the mass flow rate and volume flow rate through a cross-section with its normal inclined at an angle θ with the flow.

6. Apply mass conservation to the control volume shown enclosing a boundary layer. Is the mass flow rate $\dot{m}$ zero? If not, calculate in terms of given quantities.

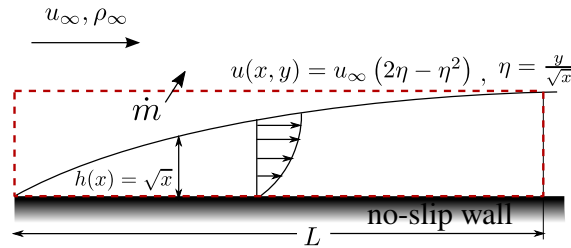


Figure 4: Boundary layer flow

7. A long flat plate of breadth L (L is small compared to the length perpendicular to the sketch) is hinged at the left side to a flat wall. The gap between the plate and the wall is filled with a liquid of density ρ . If the plate is depressed at an angular rate $\omega = d\theta/dt$. Compute the average liquid speed $u(x, t)$ with which the liquid gets squeezed out in x direction.

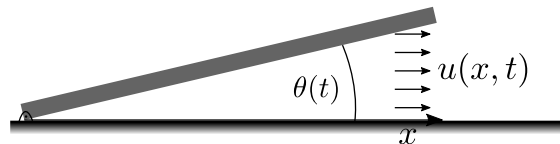


Figure 5: Hinge squeezing a flow

8. Consider a corner channel of width b (inside the page). A fluid falls uniformly into the channel at the rate of $\dot{q} \text{ m}^2/\text{s}$, which is volume flow rate per unit length. of the channel in x direction. The profile of the free surface of the fluid film developed is defined by the thickness of the fluid film h .

- (a) Develop a continuity equation involving h for the flow in the channel assuming $\dot{q}$ is constant in time and the flow is steady.

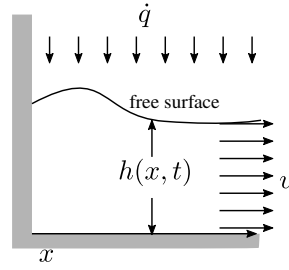


Figure 6: Free surface of a fluid film

- (b) Develop the equation for a case when $\dot{q}$ varies in time such that h is also varying in x and t . Show the appropriate control volume clearly.
9. Calculate the rate of increase of mass of air inside the room according to shown scenario. Air enters the room at a uniform velocity u_1 through area A_1 . The outlet velocity is $u_2(r) = u_0 \left(1 - \frac{r^2}{R^2}\right)$ through a cylindrical duct of radius R . The leakage happens at velocity u_3 through an area A_3 . Assume incompressible flow.

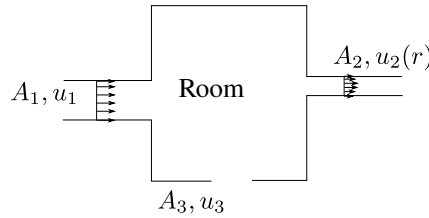


Figure 7: Air flow balance of a room

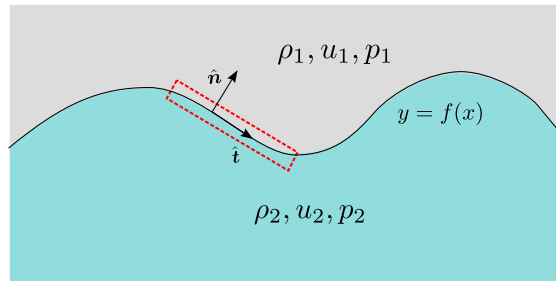


Figure 8: Interface of two fluids

10. Consider the interface shown between the two fluids of respective density, velocity, and pressures. Apply the mass balance across the shown control volume. Taking the limit of a thin control volume, determine the jump conditions on the interface using the normal and tangential vectors $\hat{n}$ and $\hat{t}$. Also, for completeness, please state what the normal and tangential vectors would be for given profile $y = f(x)$.
11. A water jet of speed u and cross-sectional area A fills a cart of mass M_0 , initially at rest as shown in figure 9.
- Derive an equation for the rate of change of mass of the cart in terms of $V_c(t)$.
 - Find $V_c(t)$.
12. Consider the wake behind the airfoil in a flow of velocity U_∞ . Far away from the wake, the measured velocity within the wake can be approximated as $u(x, y) = U_\infty - u_0 \cos\left(\frac{\pi y}{b(x)}\right)$ for $-1/2 \leq y/b(x) \leq 1/2$ by a student based on the experimental data. Determine the drag force per unit thickness into the paper on the airfoil using the control volume II of width b_0 . The curves defining the boundaries of the control volume are streamlines.

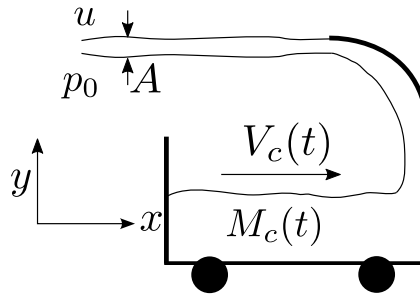


Figure 9: Cart with variable mass

Now consider the control volume I. Is there a mass flux across its control surface? If yes, calculate it in terms of variables given.

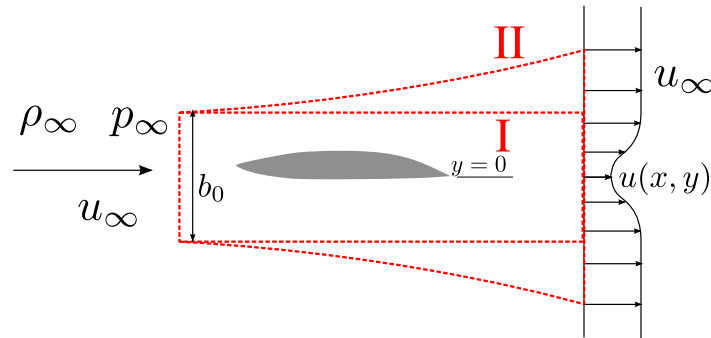


Figure 10: Drag on an airfoil

13. Water jet hits a stationary plate inclined at an angle $\theta = 60^\circ$. Assuming $v_1 = v_2 = v_3$ and that the force required to keep the plate stationary F_n acts normally,

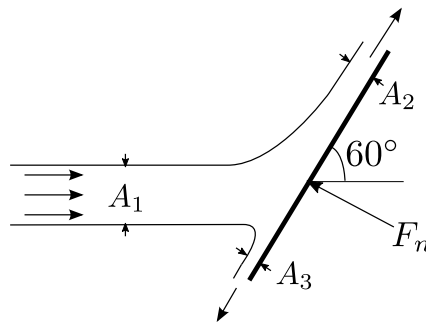


Figure 11: Stationary plate

- (a) Calculate the areas A_2 and A_3 in terms of A_1 .
 - (b) The force F_n .
14. A hydraulic jump is a sudden increase in the depth of a liquid stream. Due to this sudden increase there is a lot of mixing leading to energy loss. Consider the situation given below in which the height of the liquid stream abruptly jumps from h_1 to h_2 . Find the ratio h_2/h_1 in terms of ρ, U_1, U_2, g . Determine the mathematical condition for this hydraulic jump to occur.
15. Consider flow of air of uniform speed V_1 and density ρ past an infinite set of blades which turn the flow through an angle θ as shown. The blade spacing is S and the width of each blade is W in the direction normal to the paper. Derive expressions for,
- (a) The exit speed
 - (b) The pressure drop $p_1 - p_2$

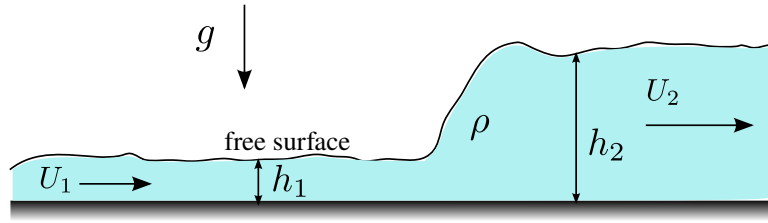


Figure 12: Hydraulic jump

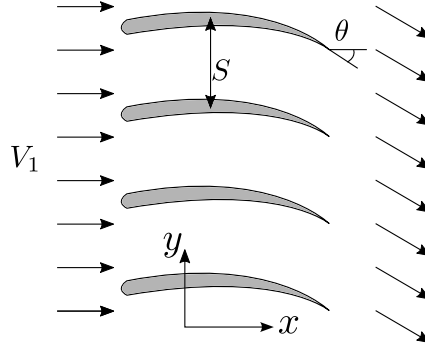


Figure 13: Blades turning flow

(c) Forces F_x and F_y that act on a single blade.

16. Water flows at $Q_1 = 0.05 \text{ m}^3/\text{s}$ through a Y-connector and splits into two streams of equal speed. The pressure at point 1 is $2 \times 10^5 \text{ Pa}$. Find the x and y components of force required to hold the connector in place. Neglect the weight of the liquid and the connector. Areas of the connected pipes are $A_1 = 0.01 \text{ m}^2$, $A_2 = 0.002 \text{ m}^2$, and $A_3 = 0.003 \text{ m}^2$.

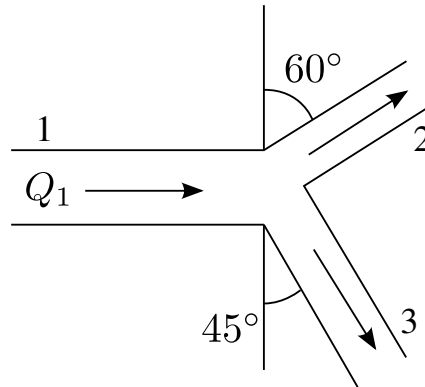


Figure 14: Y-connector

17. Consider the siphon shown in figure 15. The heights are indicated. The liquid flows out of the tube with a velocity u . Can you calculate this by applying Bernoulli's equation between points B and D? If yes, then show. If not, then which pair of points maybe used? Assume atmospheric pressure is p_0 .
18. Pipe of area A branches symmetrically into two pipes of length L and same area A at an angle of α . Water of density ρ flows at volume flow rate Q . The flow rate is divided into the branches such that $Q_1 = \eta Q$. The whole assembly is rotating at an angular velocity of ω about the z -axis as shown in figure 16

(a) Apply the angular momentum equation in the frame rotating with the assembly. (Take cross product of position vector $\vec{r}$ with the momentum equation in the rotating frame). Verify that any external moment $\vec{M}$ required to keep the assembly rotating at constant ω is given by,

$$\rho \int_V \vec{r} \times (\vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2\vec{\omega} \times \vec{v} + \dot{\vec{\omega}} \times \vec{r}) dV = \vec{M}.$$

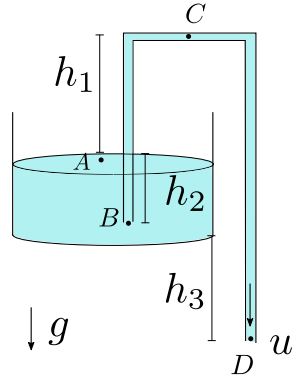


Figure 15: Siphon

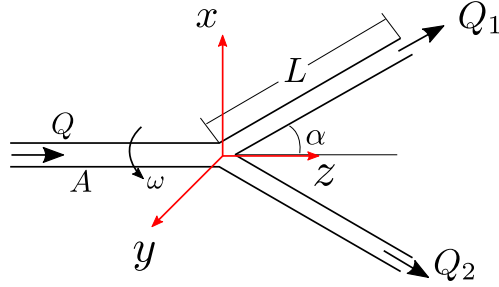


Figure 16: Rotating pipe assembly

- (b) Now apply the above equation to the shown assembly. What is the contribution of centripetal acceleration from the lower branch?
- (c) Compute the net moment required in z direction to keep the assembly rotating at constant angular velocity.

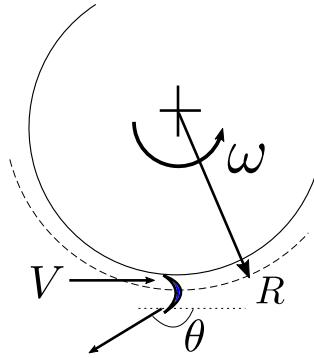


Figure 17: Pelton wheel

19. A Pelton wheel is shown in figure 17. Water jet of cross-sectional area A hits the bucket of the wheel at speed V and is turned at an angle θ with same speed. As a result the wheel rotates with an angular velocity ω . Such kind of wheels are used for generation of hydro-power in areas where sufficient water head is available. Calculate the power delivered to the wheel by the water jet. Assume the speed of the bucket as $R\omega$.
20. Consider the corner channel of width b (inside the page) from the previous problem set. A fluid falls uniformly into the channel at the rate of $\dot{q} \text{ m}^2/\text{s}$, which is volume flow rate per unit length, of the channel in x direction. The profile of the free surface of the fluid film developed is defined by the thickness of the fluid film h .
 - (a) Derive another governing equation for $v(x, t)$ and $h(x, t)$ using the momentum equation.

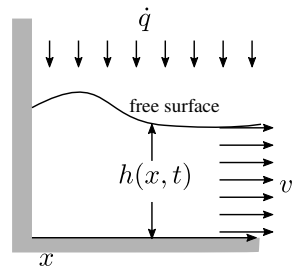


Figure 18: Free surface of a fluid film

- (b) Solve for the free surface height $h(x, t)$ for constant $\dot{q}$.