

August 12, 2026

Maximum score is 40. There are 4 questions in total. Time limit is 60 minutes. Clearly write all your assumptions and steps. No marks will be given for correct answers without any steps.

1. Simplify the following. Assume smooth and differentiable field $\underline{v}$. **(15)**

(a) $\delta_{ij}\delta_{ij}$ **(1)**

(b) $\epsilon_{ijk}\epsilon_{ijk}$ **(2)**

(c) $\nabla \cdot (\nabla r^5)$ for $r = \sqrt{x_i x_i}$ **(4)**

(d) $\nabla \cdot (\nabla \times \underline{v})$ **(4)**

(e) $\underline{v} \cdot (\nabla \times \underline{v})$ **(4)**

Answer:-

(a) $\delta_{ij}\delta_{ij} = \delta_{ii} = 3$

(b) $\epsilon_{ijk}\epsilon_{ijk} = \delta_{jj}\delta_{kk} - \delta_{jk}\delta_{kj} = 3 \cdot 3 - 3 = 6$

(c)

$$\nabla \cdot (\nabla r^5) = 5 \nabla \cdot (r^4 \nabla r) = 5 (4r^3 \nabla r \cdot \nabla r + r^4 \nabla \cdot \nabla r) = 5 (4r^3 |\nabla r|^2 + r^4 \nabla \cdot \nabla r)$$

Substituting (this has to be shown in your solution)

$$\nabla r = \frac{\underline{r}}{r}$$

$$\nabla \cdot \nabla r = \nabla \cdot \frac{\underline{r}}{r} = \frac{\nabla \cdot \underline{r}}{r} - \frac{\underline{r} \cdot \nabla r}{r^2} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

we obtain

$$5 \left(4r^3 \frac{r^2}{r^2} + r^4 \frac{2}{r} \right) = 30r^3$$

(d)

$$\nabla \cdot (\nabla \times \underline{v}) = \hat{e}_i \frac{\partial}{\partial x_i} \left(\hat{e}_j \frac{\partial}{\partial x_j} \times \hat{e}_k v_k \right) = \frac{\partial^2 v_k}{\partial x_i \partial x_j} \hat{e}_i \cdot \hat{e}_j \times \hat{e}_k = \epsilon_{ijk} \frac{\partial^2 v_k}{\partial x_i \partial x_j}$$

Swapping dummy indices and then using definitions

$$\epsilon_{ijk} \frac{\partial^2 v_k}{\partial x_i \partial x_j} = \epsilon_{jik} \frac{\partial^2 v_k}{\partial x_j \partial x_i} = -\epsilon_{ijk} \frac{\partial^2 v_k}{\partial x_i \partial x_j} = 0$$

(e)

$$\underline{v} \cdot (\nabla \times \underline{v}) = \hat{e}_i v_i \cdot \left(\hat{e}_j \frac{\partial}{\partial x_j} \times \hat{e}_k v_k \right) = v_i \frac{\partial v_k}{\partial x_j} \epsilon_{ijk}$$

Since $v_i \frac{\partial v_k}{\partial x_j} \neq v_k \frac{\partial v_i}{\partial x_j}$, the above expression is not necessarily symmetric. Hence there is no further simplification possible. $\nabla \times \underline{v}$ is not necessarily normal to $\underline{v}$ in 3D. For $\underline{v}$ being the velocity field, $\underline{v} \cdot (\nabla \times \underline{v})$ is a quantity called *helicity* of the flow. 3 marks cut if you performed incorrect forced simplification to get 0.

2. In a 2-D flow field, the velocity field is given by, **(8)**

$$\underline{v}(t) = 4\hat{i}, \quad 0 \leq t < 4 \text{ s}$$

$$4\hat{i} + 4\hat{j}, \quad 4 \leq t < 8 \text{ s}$$

$$4\hat{j}, \quad 8 \leq t < 12 \text{ s.}$$

Qualitatively sketch the following showing the origin (0,0) on each graph.

- (a) Streamline passing through $(0,0)$ at $t = 2$ s. **(1)**
- (b) Pathline of a particle which was at $(0,0)$ at $t = 0$, observed at $t = 2$ s. **(1)**
- (c) Streamline passing through the origin at $t = 10$ s. **(2)**
- (d) Pathline of the particle in part 2b observed at $t = 10$ s. **(2)**
- (e) Streakline of particles released from origin from $t = 0$ onwards, observed at $t = 10$ s. **(2)**

Answer:-

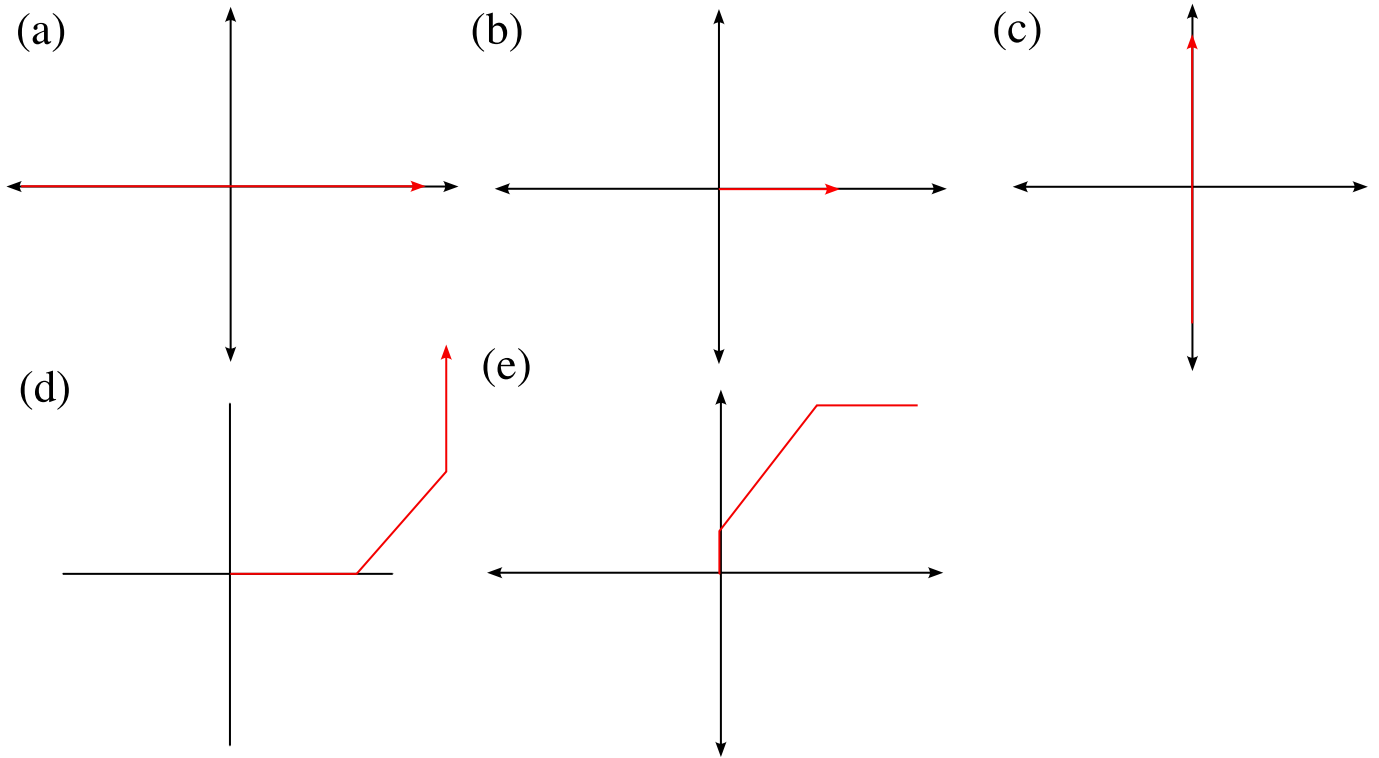


Figure 1: Problem 1 solution.

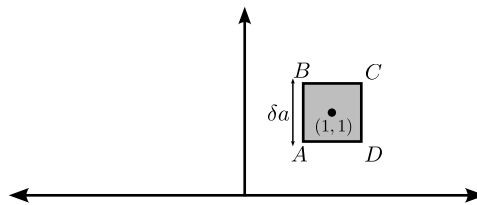


Figure 2: Fluid element $ABCD$.

3. Velocity field in a steady 2-D flow is given by **(13)**,

$$u = x, \quad v = -y.$$

- (a) Find the equation of streamlines. Sketch the streamlines of the flow in the region $-\infty < x < \infty, y > 0$. Show atleast three curves in each quadrant. **(2)**
- (b) Consider a square shaped fluid element of side length δa centered at $(1,1)$ at time t , as shown in figure 2. Tabulate the velocities of points $ABCD$. **(4)**
- (c) Calculate the acceleration of an infinitesimal fluid element when it is at $(1,1)$. **(2)**
- (d) Compute the velocity gradient tensor and decompose in symmetric and anti-symmetric parts. What is the rate of bulk strain in this flow? **(3)**
- (e) Sketch the qualitative shape of the fluid element in part 3b at time $t + \delta t$. Assume small δa . **(2)**

Answer:-

(a) Equation of streamlines,

$$\frac{dy}{dx} = \frac{v}{u}, \implies xy = \text{const.}$$

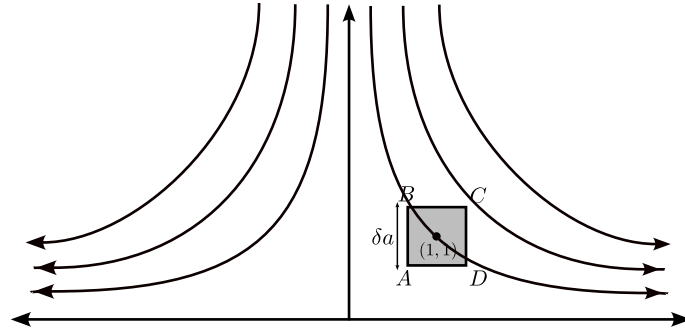


Figure 3: Streamlines

$$\begin{aligned} \text{(b)} \quad u_A &= 1 - \frac{\delta a}{2}, \quad v_A = -1 + \frac{\delta a}{2} \\ u_B &= 1 - \frac{\delta a}{2}, \quad v_B = -1 - \frac{\delta a}{2} \\ u_C &= 1 + \frac{\delta a}{2}, \quad v_C = -1 - \frac{\delta a}{2} \\ u_D &= 1 + \frac{\delta a}{2}, \quad v_D = -1 + \frac{\delta a}{2} \end{aligned}$$

(c) Acceleration is given by,

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \underline{v} \cdot \nabla v,$$

which yields,

$$\frac{du}{dt} = x$$

and

$$\frac{dv}{dt} = y.$$

(d) For 2D, velocity gradient tensor is

$$\nabla \underline{v} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The tensor is symmetric so antisymmetric part is 0.

(e) While for small δa it is ok to reason that the fluid particle retains the shape. Reasonings such as $\nabla \cdot \underline{v} = 0$ so there is no deformation are wrong. For any δa , D moves slightly more to right than A. B moves slightly

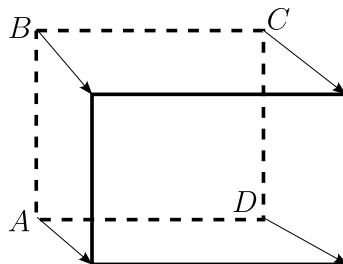


Figure 4: Deformed shape

more downward than A. C moves as downward as B and as rightward as D. The new shape is rectangular, elongated in x and compressed in y . This is clear from the strain rate components also. $\partial u/\partial x = 1$ means longitudinal stretching and $\partial v/\partial y = -1$ means longitudinal compression.

4. If S_{ij} is the symmetric part of velocity gradient tensor $\partial v_i/\partial x_j$, show that $S_{ij} \left(S_{ij} - \frac{1}{3} \nabla \cdot \underline{v} \delta_{ij} \right) \geq 0$. **(4)**

Answer:-

$$S_{ij} S_{ij} - \frac{1}{3} \nabla \cdot \underline{v} \delta_{ij} S_{ij} = S_{ij} S_{ij} + \frac{1}{3} \nabla \cdot \underline{v} \delta_{ij} S_{ij} - \frac{2}{3} \nabla \cdot \underline{v} \delta_{ij} S_{ij}$$

Since $\nabla \cdot \underline{v} = S_{ij} \delta_{ij}$ and $\delta_{ij} \delta_{ij} = 3$, we can write the above

$$S_{ij} S_{ij} + \frac{(\nabla \cdot \underline{v})^2 \delta_{ij} \delta_{ij}}{9} - \frac{2}{3} S_{ij} \nabla \cdot \underline{v} \delta_{ij} = \left(S_{ij} - \frac{\nabla \cdot \underline{v} \delta_{ij}}{3} \right)^2 \geq 0$$