

September 9, 2026

Maximum score is 40. There are 3 questions in total. Time limit is 60 minutes. Clearly write all your assumptions and steps. No marks will be given for correct answers without any steps.

1. A sealed container of radius R , filled with water of density ρ_w , is rotating about its vertical axis at a fixed angular speed of Ω . A very small spherical cork of radius $a \ll R$ and density $\rho_c < \rho_w$ is attached by a thread of length $L < R$ to the outer bottom edge of the container as shown. The container, water, and cork rotate like a rigid body. **(15)** Answer:- For rigidly rotating water inside the tank,

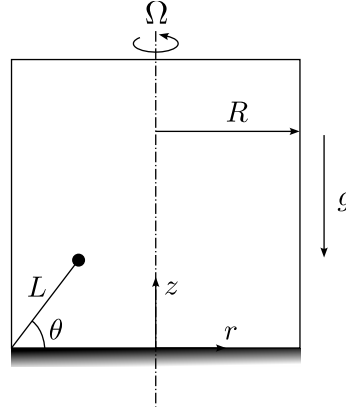


Figure 1: Rotating sealed container

$$\nabla p = \rho_w (\underline{g} - \underline{a}). \quad (0.1)$$

- (a) Water is stationary in the frame rotating with the tank at $\underline{\Omega} = \Omega \hat{k}$. Acceleration at a point r away from the axis is,

$$\underline{a} = \underline{\Omega} \times (\underline{\Omega} \times r \hat{r}) = -\Omega^2 r \hat{r}. \quad (0.2)$$

Pressure gradient is,

$$\nabla p = -\rho_w g \hat{k} + \rho_w \Omega^2 r \hat{r}. \quad (0.3)$$

- (b) Net force on a body due to fluid is,

$$\underline{F} = - \oint_S p \hat{n} dS \quad (0.4)$$

which by Gauss' divergence theorem gives,

$$\underline{F} = - \oint_S p \hat{n} dS = - \int_V \nabla p dV. \quad (0.5)$$

If the volume of the body is very small,

$$\underline{F} \approx -\nabla p V. \quad (0.6)$$

For a small cork at $r = R - L \cos \theta$ and volume $V_c = \frac{4\pi a^3}{3}$,

$$\underline{F} = \rho_w g V_c \hat{k} - \rho_w \Omega^2 r V_c \hat{r}. \quad (0.7)$$

Force balance on the cork yields

$$\begin{aligned} \rho_w g V_c &= \rho_c g V_c + T \sin \theta \\ \rho_w \Omega^2 r V_c &= \rho_c \Omega^2 r V_c + T \cos \theta \\ \implies \tan \theta &= \frac{g}{\Omega^2 r} \end{aligned}$$

It is ok to leave this in terms of $r = R - L \cos \theta$.

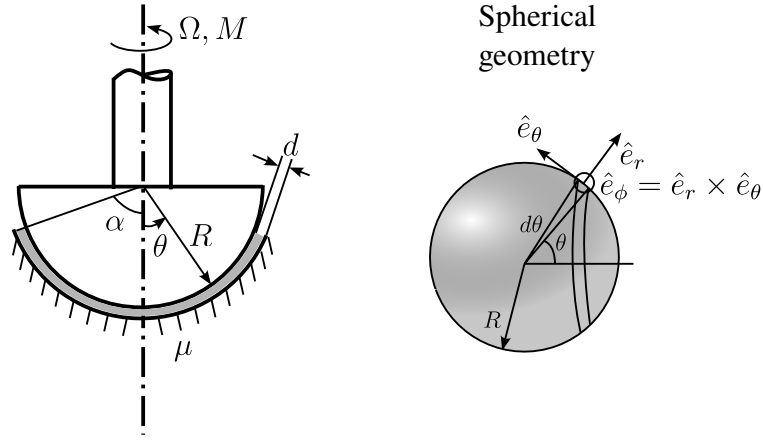


Figure 2: Rotating spherical bearing

2. A spherical thrust bearing is shown. The gap between the spherical member and the housing is constant and equal to d . Take the oil between the bearing and housing to have viscosity μ . Take help of the spherical geometry figure shown on the right.

- (a) Assume uni-directional flow with velocity varying linearly as $Ar + B$. Using appropriate boundary conditions, derive A and B . Specify the non-zero component of velocity **(3)**

Answer:- The velocity in spherical coordinates is

$$\underline{v} = u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_\phi \hat{e}_\phi. \quad (0.8)$$

Since the flow is rotational, only u_ϕ will be non-zero. For $u_\phi = Ar + B$, $u_\phi = -R \sin \theta \Omega$ at $r = R$ and $u_\phi = 0$ at $r = R + d$. This yields,

$$u_\phi = -R \Omega \sin \theta \left(1 + \frac{R}{d} - \frac{r}{d} \right) \quad (0.9)$$

- (b) What is the unit normal of the upper surface? Use that to identify the stress tensor components contributing to the surface traction. Compute the traction vector **(5)**

Answer:- Unit normal is $\hat{e}_r$. The traction force will be $\underline{t} = \underline{\underline{\sigma}} \cdot \hat{n} = \sigma_{rr} \hat{e}_r + \sigma_{r\theta} \hat{e}_\theta + \sigma_{r\phi} \hat{e}_\phi$ where components are evaluated at $r = R$. From the formulae given

$$\sigma_{rr} = \sigma_{r\theta} = 0. \quad (0.10)$$

And

$$\sigma_{r\phi} = \mu r \frac{\partial}{\partial r} \left(\frac{u_\phi}{r} \right) = -\mu \frac{B}{R} = \mu \Omega \sin \theta \left(1 + \frac{R}{d} \right). \quad (0.11)$$

The traction vector is $\underline{t} = \mu \Omega \sin \theta \left(1 + \frac{R}{d} \right) \hat{e}_\phi$.

- (c) Integrate the torque due to viscous shear stress to obtain a relation between M , μ , Ω , and geometric parameters. **(7)**

Answer:- The area element is $dS = R^2 \sin \theta d\phi d\theta$ for the spherical coordinates. Writing the moment on this,

$$- \int \sigma_{r\phi} R \sin \theta dS = \int_0^{2\pi} d\phi \int_0^\alpha d\theta \mu \Omega R^3 \left(1 + \frac{R}{d} \right) \sin^3 \theta = 2\pi \mu \Omega R^3 \left(1 + \frac{R}{d} \right) \int_0^\alpha \sin^3 \theta d\theta \quad (0.12)$$

For steady state,

$$M = 2\pi \mu \Omega R^3 \left(1 + \frac{R}{d} \right) \left(\frac{\cos^3 \alpha}{3} - \cos \alpha + \frac{2}{3} \right) \quad (0.13)$$

3. Water-air interface near a plane wall makes a shape $\eta(x)$ as shown. Ignore air density. Density of water is ρ and surface tension coefficient of the interface is σ . Wetting angle at $x = 0$ is known to be θ . For $x \rightarrow \infty$, a horizontal surface is observed.

- (a) Using $-\frac{1}{2}\nabla \cdot \hat{n}$ expression for the mean curvature, assuming planar shape in depthwise direction, derive an expression for the mean curvature. **(5)**

Answer:- As covered in class,

$$p_a - p_w = 2H\sigma. \quad (0.14)$$

For $\hat{n} = \frac{\nabla(y - \eta(x))}{|\nabla(y - \eta(x))|}$, we obtain

$$\hat{n} = \frac{\hat{j} - \eta' \hat{i}}{\sqrt{1 + \eta'^2}}. \quad (0.15)$$

Hence, the mean curvature is

$$H = -\frac{1}{2}\nabla \cdot \hat{n} = -\frac{1}{2} \frac{-\eta''}{(1 + \eta'^2)^{3/2}} \quad (0.16)$$

- (b) Assuming small slope, approximate the mean curvature expression above. Compute the maximum height h at the wall. **(5)**

Answer:- For small slope,

$$p_a - p_w \approx \sigma \eta''. \quad (0.17)$$

Since $p_w = p_a$ as the interface becomes flat, at the curved interface, we have $p_w = p_a - \rho g \eta$ which yields,

$$\rho g \eta = \sigma \eta'' \quad (0.18)$$

which yields the solution

$$\eta = C e^{-\sqrt{\frac{\rho g}{\sigma}} x}. \quad (0.19)$$

At $x = 0$, we have $\eta' = -\cot \theta$ which yields

$$C = \frac{\sigma}{\rho g} \cot \theta = \eta(x = 0) \quad (0.20)$$

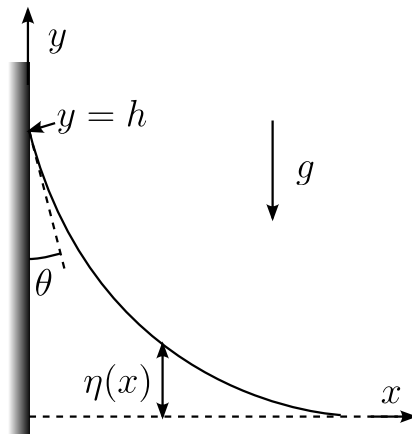


Figure 3: Air water interface

Useful equations

- Radial component of momentum equation in cylindrical coordinates: $\rho \left(\frac{\partial u_r}{\partial t} + \underline{v} \cdot \nabla u_r - \frac{u_\theta^2}{r} \right) = -\frac{\partial p}{\partial r} + \mu \left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right) + f_r$ for $\underline{v} = u_r \hat{e}_r + u_\theta \hat{e}_\theta + u_z \hat{e}_z$.
- Viscous stress components in spherical coordinates

$$1. \sigma_{rr} = 2\mu \frac{\partial u_r}{\partial r}$$

$$2. \sigma_{\theta\theta} = 2\mu \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_\theta}{r} \right)$$

3. $\sigma_{\phi\phi} = 2\mu \left(\frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta \cot \theta}{r} \right)$
4. $\sigma_{\theta\phi} = \mu \left(\frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{u_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \phi} \right) = \sigma_{\phi\theta}$
5. $\sigma_{\phi r} = \mu \left(\frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \phi} + r \frac{\partial}{\partial r} \left(\frac{u_\phi}{r} \right) \right) = \sigma_{r\phi}$
6. $\sigma_{r\theta} = \mu \left(r \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right) + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) = \sigma_{\theta r}$