AML2020/APL107/APL106 (2026-2027-I): Quiz-5-solutions

Department of Applied Mechanics, IIT Delhi

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Maximum score is 40. There are 2 questions in total. Time limit is 60 minutes. Clearly write all your assumptions and steps. No marks will be given for correct answers without any steps.

1. We may simulate potential flow of speed U_∞ past a two-dimensional ridge or bump of height $a/2$ by using a streamline that passes above the potential flow over a cylinder as shown below.

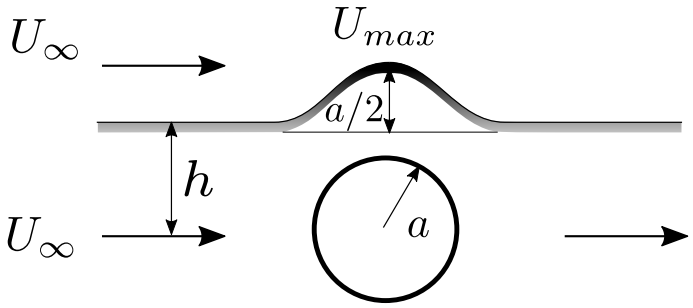


Figure 1: Potential flow past a bump

(a) Write the streamfunction ψ and velocity potential ϕ for potential flow past a cylinder of radius a in polar coordinates. **(2)**

Answer:- For a uniform flow

$$\phi_{\text{uniform}} = U_\infty x = U_\infty r\cos\theta,\ \psi_{\text{uniform}} = U_\infty y = U_\infty r\sin\theta$$

For a doublet,

$$\phi_{\text{doublet}} = -\frac{\Lambda\cos\theta}{r},\ \psi_{\text{doublet}} = -\frac{\Lambda\sin\theta}{r}$$

Adding them together,

$$\phi = U_\infty r\cos\theta - \frac{\Lambda\cos\theta}{r},\ \psi = U_\infty r\sin\theta - \frac{\Lambda\sin\theta}{r}$$

Radial and azimuthal velocity are

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U_\infty\cos\theta - \frac{\Lambda\cos\theta}{r^2},\ u_\theta = -\frac{\partial\psi}{\partial r} = -U_\infty\sin\theta - \frac{\Lambda\sin\theta}{r^2} \tag{0.1}$$

$u_r = 0$ at $r = a = \sqrt{\Lambda/U_\infty} \implies \Lambda = U_\infty a^2$

(b) Using ψ, calculate the height h corresponding to the streamline which has a deflection of $a/2$ due to the cylinder. **(8)**

Answer:- At $r = a$ for $\Lambda = U_\infty a^2$, $\psi = 0$. Since $\psi_2 - \psi_1$ between two streamlines is the flow rate. $\int \frac{\partial\psi}{\partial y}dy = U_\infty h = \psi_2 - \psi_1$. $\psi = U_\infty h$ at the streamline h above the central streamline. Passes through $r = h + \frac{a}{2}$ and $\theta = \frac{\pi}{2}$.

$$\psi = U_\infty h = U_\infty\left(h + \frac{a}{2}\right) - \frac{U_\infty a^2}{h + \frac{a}{2}} \implies U_\infty h\left(h + \frac{a}{2}\right) = U_\infty\left(h + \frac{a}{2}\right)^2 - U_\infty a^2$$

From the above, we get,

$$\frac{U_\infty h a}{2} = -\frac{3U_\infty a^2}{4} + U_\infty a h \implies h = \frac{3a}{2}$$

(c) Assuming the streamline of the above part (b) as a solid surface with the desired bump, calculate the maximum velocity U_{max} at the top of the bump. **(5)**
Answer:- At $\theta = \pi/2$, $u_r = 0$. For u_θ at $r = 2a$, $\theta = \pi/2$

$$u_\theta = -\frac{5U_\infty}{4}$$

(d) Calculate the pressure distribution and the pressure coefficient $c_p = \dfrac{(p - p_\infty)}{\frac{1}{2}\rho U_\infty^2}$ over the bump. **(5)**

Answer:- From Bernoulli's equation,

$$p_\infty - p = \frac{\rho U_\infty^2}{2}\left(1 - \frac{u_r^2 + u_\theta^2}{U_\infty^2}\right)$$

The bump's points satisfy

$$\frac{3a}{2} = r\sin\theta - \frac{a^2\sin\theta}{r} \implies \sin\theta = \frac{3ar}{2(r^2 - a^2)}$$

Substituting $u_r = U_\infty \cos\theta\left(1 - \dfrac{a^2}{r^2}\right)$, $u_\theta = -U_\infty \sin\theta\left(1 + \dfrac{a^2}{r^2}\right)$,

$$c_p = 1 - \left(\cos^2\theta\left(1 - \frac{a^2}{r^2}\right)^2 + \sin^2\theta\left(1 + \frac{a^2}{r^2}\right)^2\right)$$

Simplifying

$$c_p = 1 - \left(1 + \frac{a^4}{r^4} - \frac{2a^2}{r^2}\cos 2\theta\right) = \frac{2a^2}{r^2}\cos 2\theta - \frac{a^4}{r^4}$$

2. For a flat plate laminar boundary layer, assuming $\dfrac{u}{U_\infty} = a + b\eta + c\eta^2$ for $\eta = \dfrac{y}{\delta(x)}$.

(a) Write three boundary conditions for u with $\delta(x)$ being the boundary layer thickness. **(3)**
Answer:- $u = 0$ at $\eta = 0$, $u = U_\infty$ at $\eta = 1$. Using $u''(y = 0) = 0$ will give $c = 0$ (which is fine). Otherwise, $u'(\eta = 1) = 0$ is also a possibile condition. Using that,

$$\frac{u}{U_\infty} = 2\eta - \eta^2$$

(b) Compute the momentum thickness Θ and the displacement thickness δ^*. **(3)**
Answer:- With $u/U_\infty = 2\eta - \eta^2$ and $dy = \delta\, d\eta$:

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U_\infty}\right)dy = \delta\int_0^1 (1 - 2\eta + \eta^2)\, d\eta = \frac{\delta}{3},$$

$$\Theta = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)dy = \delta\int_0^1 (2\eta - \eta^2)(1 - 2\eta + \eta^2)\, d\eta = \frac{2\delta}{15}.$$

(c) Using the boundary layer equation, derive the Von-Karmann momentum integral equation. **(5)**
Answer:- The steady boundary-layer momentum equation is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_\infty\frac{dU_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2}.$$

Integrate across the layer from $y = 0$ to $y = \delta$ and use continuity $v = -\displaystyle\int_0^y \frac{\partial u}{\partial x}dy'$ with $\tau_w = \mu\, \partial u/\partial y|_0$.
This gives the von Kármán momentum integral equation

$$\frac{\tau_w}{\rho U_\infty^2} = \frac{d\Theta}{dx} + (2\Theta + \delta^*)\frac{1}{U_\infty}\frac{dU_\infty}{dx}.$$

For a flat plate ($dU_\infty/dx = 0$), this reduces to

$$\tau_w = \rho U_\infty^2 \frac{d\Theta}{dx}.$$

(d) Substitute the quadratic relation in the momentum integral and compute $\dfrac{\delta}{x}$. **(5)**

Answer:- Wall shear for the quadratic profile: $\tau_w = \mu \dfrac{\partial u}{\partial y}\Big|_0 = \dfrac{\mu U_\infty}{\delta} \dfrac{d}{d\eta}(2\eta - \eta^2)\Big|_0 = \dfrac{2\mu U_\infty}{\delta}$. With $\Theta = \dfrac{2\delta}{15}$ in $\tau_w = \rho U_\infty^2 \, d\Theta/dx$:

$$\frac{2\mu U_\infty}{\delta} = \frac{2}{15}\rho U_\infty^2 \frac{d\delta}{dx} \;\Rightarrow\; \delta \, d\delta = \frac{15\nu}{U_\infty} dx \;\Rightarrow\; \delta^2 = \frac{30\nu x}{U_\infty},$$

$$\frac{\delta}{x} = \frac{\sqrt{30}}{\sqrt{\mathrm{Re}_x}} = \frac{5.48}{\sqrt{\mathrm{Re}_x}}.$$

(e) Derive the skin-friction relation $c_f(x)$ and the drag coefficient assuming a plate of length L and width b. **(4)**

Answer:- $c_f = \dfrac{\tau_w}{\frac{1}{2}\rho U_\infty^2} = \dfrac{2(2\mu U_\infty/\delta)}{\rho U_\infty^2} = \dfrac{4\nu}{U_\infty \delta}$; substituting $\delta = \sqrt{30\,\nu x/U_\infty}$:

$$c_f(x) = \frac{4}{\sqrt{30}} \frac{1}{\sqrt{\mathrm{Re}_x}} = \frac{0.730}{\sqrt{\mathrm{Re}_x}}.$$

Drag coefficient (one side, length L):

$$C_D = \frac{1}{L}\int_0^L c_f \, dx = \frac{8}{\sqrt{30}} \frac{1}{\sqrt{\mathrm{Re}_L}} = \frac{1.46}{\sqrt{\mathrm{Re}_L}}, \qquad F = C_D \, \tfrac{1}{2}\rho U_\infty^2 \, bL.$$