

AML7870 (2025-2026-II): Mid-term exam solutions

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Maximum score is 60, time limit is 120 minutes. There are 5 problems in total. Clearly write all your assumptions and steps. No marks will be given for correct answers without any steps.

1. Derive the form of the equality constraint $f(x, y) = 0$ so that it is convex. Both $x, y \in \mathbb{R}$. **(5)**

Answer:- Let $\underline{r} = (x, y)$. For convexity of an inequality constraint,

$$f(\lambda_1 \underline{r}_1 + \lambda_2 \underline{r}_2) \leq \lambda_1 f(\underline{r}_1) + \lambda_2 f(\underline{r}_2). \quad (1)$$

If the constraint $f(\underline{r}) = 0$ is convex, then the constraints $f \leq 0$ and $-f \leq 0$ should both be convex. Note that we are not talking about convexity of the function here, but convexity of the set of $\underline{r}$ which satisfy the constraint. Hence, both,

$$f(\lambda_1 \underline{r}_1 + \lambda_2 \underline{r}_2) \leq \lambda_1 f(\underline{r}_1) + \lambda_2 f(\underline{r}_2) \quad (2)$$

and

$$f(\lambda_1 \underline{r}_1 + \lambda_2 \underline{r}_2) \geq \lambda_1 f(\underline{r}_1) + \lambda_2 f(\underline{r}_2) \quad (3)$$

hold simultaneously. Note that this is possible only if

$$f(\lambda_1 \underline{r}_1 + \lambda_2 \underline{r}_2) = \lambda_1 f(\underline{r}_1) + \lambda_2 f(\underline{r}_2) \implies f(\underline{r}) = \underline{c}^T \underline{r}, \quad (4)$$

for some constant $\underline{c}$.

2. Consider a three-bar truss member as shown. Derive the equilibrium equation for the truss in the form of $\underline{\underline{K}} \underline{u} = \underline{F}$ where $\underline{u}$ consists of all the displacement degrees of freedom and $\underline{F}$ contains the force components along those degrees of freedom. **(10)**

Answer:- As covered in class, for a truss member with angle θ from the horizontal and $\hat{e}$ a unit vector from node 1 to node 2 of the member, the following relations hold,

- Displacement vector : $\underline{u} = [\underline{u}_1^T, \underline{u}_2^T]^T$ where $\underline{u}_1$ and $\underline{u}_2$ are displacements of nodes 1 and 2.
- Elongation : $\delta = \underline{\underline{B}} \underline{u}$ where $\underline{\underline{B}} = [-\hat{e}^T, \hat{e}^T]$.
- Force : $\underline{f} = \underline{\underline{B}}^T \frac{EA}{l} \delta = \underline{\underline{B}}^T D \underline{\underline{B}} \underline{u} = \underline{\underline{k}} \underline{u}$
- For each bar, we can write $\underline{u} = \underline{\underline{C}} \underline{U}$, where $\underline{U}$ is the global degrees of freedom vector.

For the specific problem, the overall displacement degrees of freedom are

$$\underline{U} = [u_{3x}, u_{3y}, u_{1y}]^T. \quad (5)$$

Hence, for each bar, we can write $\underline{u} = \underline{\underline{C}} \underline{U}$,

$$\underline{u}_b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underline{U}, \quad \underline{u}_c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \underline{U}, \quad \underline{u}_a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \underline{U} \quad (6)$$

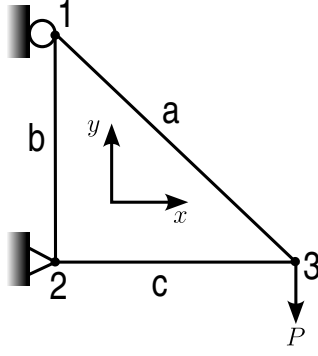
with the unit vectors

$$\hat{e}_a = [\cos \theta, -\sin \theta]^T, \hat{e}_b = [0, 1]^T, \hat{e}_c = [1, 0]^T \quad (7)$$

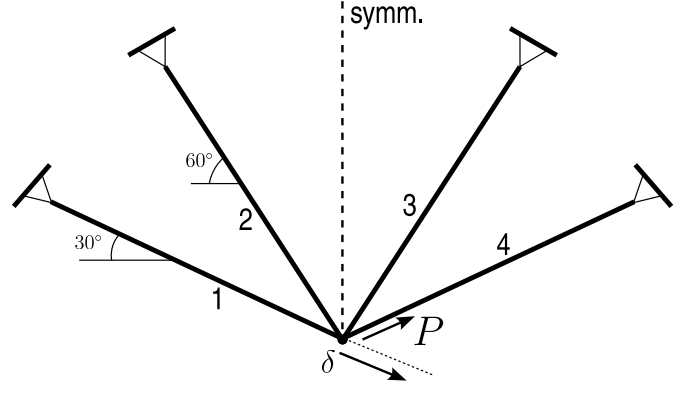
Assemble $\underline{\underline{k}}_j = \underline{\underline{B}}_j^T D_j \underline{\underline{B}}_j$ for $j = a, b, c$. And then compute

$$\underline{\underline{K}}_j = \underline{\underline{C}}_j^T \underline{\underline{k}}_j \underline{\underline{C}}_j \quad (8)$$

and $\underline{\underline{K}} = \sum \underline{\underline{K}}_j$. Full points for computing all results.



Problem 2



Problem 3

3. The volume of the four-bar truss needs to be minimized under the constraint that the elongation of bar 1, $\delta \leq \delta_0$. The external force P is positive. All the bars have length ℓ and Young's modulus E . Also, the cross-sectional areas satisfy the relations $A_3 = A_2$ and $A_4 = A_1$. The design variables are the cross-sectional areas A_1 and A_2 . Define $A_0 = P\ell/(10\delta_0 E)$. The bounds on areas are $0.2A_0 \leq A_i \leq 2.5A_0$. Formulate the optimization problem. Clearly write the objective function and constraints in terms of A_1 and A_2 . **(15)**

Answer:- Let $x_{1,2} = A_{1,2}/A_0$. Let s_i be the forces in each member (positive sign implies tensile force). Let $\hat{e}_i$ be unit vectors from support points to point under load. By force balance,

$$P \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix} = [\hat{e}_1 \ \hat{e}_2 \ \hat{e}_3 \ \hat{e}_4] \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \underline{\underline{B}} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}. \quad (9)$$

Similarly, extensions $\delta_1, \delta_2, \delta_3, \delta_4$ in the bars can be related to displacement of the point as

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} = \begin{bmatrix} \hat{e}_1^T \\ \hat{e}_2^T \\ \hat{e}_3^T \\ \hat{e}_4^T \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \underline{\underline{B}}^T \begin{bmatrix} u_x \\ u_y \end{bmatrix} \quad (10)$$

Also, from the constitutive relation,

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \frac{E}{\ell} \text{diag}[A_1, A_2, A_3, A_4] \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} = \frac{E}{\ell} \text{diag}[A_1, A_2, A_3, A_4] \underline{\underline{B}}^T \begin{bmatrix} u_x \\ u_y \end{bmatrix}. \quad (11)$$

Thus, we obtain

$$\frac{E}{\ell} \underline{\underline{B}} \text{diag}[A_1, A_2, A_3, A_4] \underline{\underline{B}}^T \begin{bmatrix} u_x \\ u_y \end{bmatrix} = P \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix} = \frac{E}{\ell} \begin{bmatrix} \frac{3A_1 + A_2}{2} & 0 \\ 0 & \frac{A_1 + 3A_2}{2} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \quad (12)$$

which yields,

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \frac{P\ell}{E} \begin{bmatrix} \frac{\sqrt{3}}{A_1 + 3A_2} \\ \frac{1}{3A_1 + A_2} \end{bmatrix}. \quad (13)$$

The constraint $\delta \leq \delta_0$ yields,

$$\delta_1 = \frac{\sqrt{3}u_x}{2} - \frac{u_y}{2} = \frac{P\ell}{E} \left(\frac{3}{2(A_1 + 3A_2)} - \frac{1}{2(3A_1 + A_2)} \right) \leq \delta_0 \quad (14)$$

or

$$\frac{3}{2(x_1 + 3x_2)} - \frac{1}{2(3x_1 + x_2)} \leq 0.1 \quad (15)$$

The objective function is simply $x_1 + x_2$ with above constraint and the box constraints $0.2 \leq x_{1,2} \leq 2.5$

4. Consider the truss compliance minimization problem as covered in class.

(a) Compute the derivative of the compliance $C(\underline{x})$ w.r.t. x_j , the cross-sectional area of the j^{th} member of the truss. **(5)**

Answer:- $\frac{\partial C}{\partial x_j} = \underline{F}^T \frac{\partial \underline{u}}{\partial x_j} = -\underline{u}^T \underline{K} \underline{K}^{-1} \frac{\partial \underline{K}}{\partial x_j} \underline{u}$. The derivative of the stiffness matrix can be computed using the relation derived in class,

$$\underline{K} = \sum_j x_j \underline{K}_j^0. \quad (16)$$

Hence, $\frac{\partial C}{\partial x_j} = -\underline{u}^T \underline{K}_j^0 \underline{u}$. In terms of displacement vector of j^{th} truss member nodes, $\underline{u}_j$, we get,

$$\frac{\partial C}{\partial x_j} = -\underline{u}_j^T \underline{k}_j^0 \underline{u}_j. \quad (17)$$

(b) Using the Lagrangian $\mathcal{L}$ of the compliance minimization problem under the volume constraint $\sum_i x_i \ell_i \leq V_0$, and the box constraints $x_i^{\min} \leq x_i \leq x_i^{\max}$, derive the bounds on stress of bars. **(5)**

Answer:- The Lagrangian is given by

$$\mathcal{L} = \underline{F}^T \underline{u} + \lambda \left(\sum_i x_i \ell_i - V_0 \right), \quad (18)$$

where $\lambda \geq 0$. Differentiating the Lagrangian, we obtain

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\partial C}{\partial x_j} + \lambda \ell_j = -\underline{u}_j^T \underline{k}_j^0 \underline{u}_j + \lambda \ell_j = -\frac{\sigma_j^2 \ell_j}{E} + \lambda \ell_j \quad (19)$$

Given box constraints $x_i^{\min} \leq x_i \leq x_i^{\max}$, and assuming x_j^* that minimizes $\mathcal{L}$,

$$\sigma_j^2 \leq \frac{\lambda^*}{E}, \text{ if } x_j^* = x_j^{\min}, \quad (20)$$

$$\sigma_j^2 = \frac{\lambda^*}{E}, \text{ if } x_j^{\min} \leq x_j^* \leq x_j^{\max}, \quad (21)$$

$$\sigma_j^2 \geq \frac{\lambda^*}{E}, \text{ if } x_j^* = x_j^{\max}. \quad (22)$$

In the above, λ^* maximizes the dual function $\phi(\lambda)$ obtained by minimizing $\mathcal{L}$.

- (c) Use the MMA approximation for the compliance. Formulate the approximate Lagrangian using the volume constraint. Derive the dual function $\phi(\lambda)$ for $\lambda \geq 0$. **(5)**

Answer:- The MMA approximation of the compliance is

$$C^{M,k}(\underline{x}) = C(\underline{x}^k) - \sum_j (x_j^k - L_j^k) \underline{u}_j^{k,T} \underline{k}_j^0 \underline{u}_j^k + \sum_j \frac{(x_j^k - L_j^k)^2}{x_j - L_j^k} \underline{u}_j^{k,T} \underline{k}_j^0 \underline{u}_j^k, \quad (23)$$

using which, we get the approximate Lagrangian as,

$$\mathcal{L}^k(\underline{x}, \lambda) = C^{M,k}(\underline{x}) + \lambda \left(\sum_j \ell_j x_j - V_0 \right) = r_0^k + \sum_j \left(\frac{(x_j^k - L_j^k)^2}{x_j - L_j^k} \underline{u}_j^{k,T} \underline{k}_j^0 \underline{u}_j^k + \lambda \ell_j x_j \right) - \lambda V_0 \quad (24)$$

The dual function is obtained by

$$\phi(\lambda) = r_0^k - \lambda V_0 + \sum_j \min_{\alpha_j^k \leq x_j \leq x_j^{\max}} \left(\frac{(x_j^k - L_j^k)^2}{x_j - L_j^k} \underline{u}_j^{k,T} \underline{k}_j^0 \underline{u}_j^k + \lambda \ell_j x_j \right) \quad (25)$$

- (d) Show that $-\phi(\lambda)$ is a convex function. **(10)**

Answer:- The minimizer $\underline{x}^*$ of the function in parenthesis has components, either one of the two box constraints or,

$$x_j^* = L_j^k + (x_j^k - L_j^k) \sqrt{\frac{\underline{u}_j^{k,T} \underline{k}_j^0 \underline{u}_j^k}{\lambda \ell_j}}. \quad (26)$$

It is sufficient to prove that for each j , the function

$$\phi_j(\lambda) = \min_{\alpha_j^k \leq x_j \leq x_j^{\max}} \left(\frac{(x_j^k - L_j^k)^2}{x_j - L_j^k} \underline{u}_j^{k,T} \underline{k}_j^0 \underline{u}_j^k + \lambda \ell_j x_j \right) \quad (27)$$

is convex. For $x_j = x_j^{\max}$ or $x_j^{\min}$, the function is linear in λ and easily shown is convex. For $x_j = x_j^*$, the function is

$$\phi_j(\lambda) = 2\sqrt{q_{0j}^k \lambda \ell_j} + \lambda \ell_j L_j^k. \quad (28)$$

The linear part of ϕ_j can be again easily shown to be convex. Let $g(\lambda) = -m\sqrt{\lambda}$ for some $m > 0$. For $g(\lambda)$ to be convex,

$$g(a\lambda_1 + (1-a)\lambda_2) = -m\sqrt{a\lambda_1 + (1-a)\lambda_2} \leq ag(\lambda_1) + (1-a)g(\lambda_2), \quad (29)$$

where $0 < a < 1$. Consider the inequality

$$a\sqrt{\lambda_1} + (1-a)\sqrt{\lambda_2} \leq \sqrt{a\lambda_1 + (1-a)\lambda_2}. \quad (30)$$

Since $x_1 > x_2 \implies x_1^2 > x_2^2$ for all real x_1 and x_2 , hence,

$$a^2\lambda_1 + (1-a)^2\lambda_2 + 2a(1-a)\sqrt{\lambda_1\lambda_2} \leq a\lambda_1 + (1-a)\lambda_2 \quad (31)$$

$$a(a-1)\lambda_1 + (1-a)(-a)\lambda_2 + 2a(1-a)\sqrt{\lambda_1\lambda_2} \leq 0. \quad (32)$$

Since $0 < a < 1$, $a(a-1) < 0$, we obtain,

$$\lambda_1 + \lambda_2 - 2\sqrt{\lambda_1\lambda_2} \geq 0 \implies (\sqrt{\lambda_1} - \sqrt{\lambda_2})^2 \geq 0 \quad (33)$$

which always holds. Hence $g(\lambda)$ is a convex function. Hence $\phi_j(\lambda)$ is a concave function implying $\phi(\lambda)$ is concave.

5. For a quadratic function $f(\underline{x}) = \frac{1}{2} \underline{x}^T \underline{\underline{H}} \underline{x} - \underline{b}^T \underline{x} + c$, derive the value of the optimum step size α_k at step $k \rightarrow k+1$ defined using,

$$\underline{x}^{k+1} = \underline{x}^k + \alpha_k \underline{d}^k,$$

where $\underline{d}^k$ is the steepest descent direction at $\underline{x}^k$. **(5)**

Answer:- Choose α_k such that $\frac{\partial f(\underline{x}^{k+1})}{\partial \alpha_k} = 0$. Hence

$$\nabla f(\underline{x}^{k+1}) \cdot \frac{\partial \underline{x}^{k+1}}{\partial \alpha_k} = 0 \quad (34)$$

Substituting the function derivative (assuming symmetric $\underline{\underline{H}}$. For general $\underline{\underline{H}}$, replace $\underline{\underline{H}}$ by $\frac{\underline{\underline{H}} + \underline{\underline{H}}^T}{2}$),

$$[\underline{\underline{H}} (\underline{x}^k + \alpha_k \underline{d}^k) - \underline{b}] \cdot \underline{d}^k = 0 \implies \alpha_k \underline{d}^{k,T} \underline{\underline{H}} \underline{d}^k + \underline{d}^{k,T} \underline{\underline{H}} \underline{x}^k = \underline{d}^{k,T} \underline{b}, \quad (35)$$

from which we get (using $\underline{d}^k = \nabla f(\underline{x}^k) = \underline{\underline{H}} \underline{x}^k - \underline{b}$),

$$\alpha_k = \frac{\underline{d}^{k,T} (\underline{b} - \underline{\underline{H}} \underline{x}^k)}{\underline{d}^{k,T} \underline{\underline{H}} \underline{d}^k} = -\frac{|\nabla f(\underline{x}^k)|^2}{\underline{d}^{k,T} \underline{\underline{H}} \underline{d}^k} \quad (36)$$