Combinatorics and Graph Theory I, Spring 2019, sheet 1

Each exercise sheet gives 10 points. These points are given for:

- correctness of answers,
- conciseness of arguments, and
- readability of write-ups.

To pass the tutorial and get your credits you need a total of at least 60% of all points from exercise sheets, and 60% of all points from the two quizzes (each gives 10 points).

Please hand in your solution to this sheet at the beginning of the next tutorial on Thursday February 28th.

Exercise 1.1
Here are two different definitions for the big-Oh notation. Assume that the functions f and g are non-negative and are defined for every natural number.

Definition 1. $f(n) = \mathcal{O}(g(n))$ if there exists a constant C such that for all $n \in \mathbb{N}$, the inequality $f(n) \leq C \cdot g(n)$ holds.

Definition 2. $f(n) = \mathcal{O}(g(n))$ if there exist constants n_0 and C such that for all $n \geq n_0$, the inequality $f(n) \leq C \cdot g(n)$ holds.

(i) Show that the above two definitions are equivalent.

(ii) For each of the following, decide whether they are true or not. Give reasons to support your answer.

(a) $n^2 = \mathcal{O}(n^2 \ln n)$
(b) $n^2 = o(n^2 \ln n)$
(c) $n^2 + 5n \ln n = n^2(1 + o(1)) \sim n^2$
(d) $n^2 + 5n \ln n = n^2 + \mathcal{O}(n)$
(e) $\sum_{i=1}^{n} i^8 = \Theta(n^9)$
(f) $\sum_{i=1}^{n} \sqrt{i} = \Theta(n^{3/2})$

(iii) (Bonus question) Find two nondecreasing functions f and g defined for all natural numbers such that neither $f(n) = \mathcal{O}(g(n))$, nor $g(n) = \mathcal{O}(f(n))$.

Exercise 1.2
Let $x_1, x_2, \ldots, x_n$ be positive reals. Their *arithmetic mean* equals $(x_1 + x_2 + \cdots + x_n)/n$, and their *geometric mean* is defined as $\sqrt[n]{x_1 x_2 \cdots x_n}$. Let $AG(n)$ denote the statement "for any n-tuple of positive reals $x_1, x_2, \cdots, x_n$, the geometric mean is less than or equal to the arithmetic mean". Prove the validity of $AG(n)$ for every n by the following strange induction:

(i) Prove that $AG(n)$ implies $AG(2n)$, for each n.

(ii) Prove that $AG(n)$ implies $AG(n-1)$, for each $n > 1$.

(iii) Explain why proving (i) and (ii) is enough to prove the validity of $AG(n)$ for all n.

Exercise 1.3

(i) Show that product of all primes p with $m < p \leq 2m$ is at most $\binom{2m}{m}$.

(ii) Using (i), prove the estimate $\pi(n) = \mathcal{O}(n/\ln n)$ where $\pi(n)$ is the number of primes not greater than n.