Combinatorics and Graph Theory I, Spring 2019, Sheet 10

Each exercise sheet gives 10 points. These points are given for:

- correctness of answers,
- conciseness of arguments, and
- readability of write-ups.

To pass the tutorial and get your credits you need a total of at least 60% of all points from exercise sheets, and 60% of all points from the two quizzes (each gives 10 points).

Please hand in your solution to this sheet at the beginning of the next tutorial on Thursday, May 9th.

Exercise 10.1

(a) Prove that the $n \times n$ array L whose (i, j)th-entry is defined by

$$L(i, j) = i + j \pmod n$$

is a Latin square.

(b) Let p be a prime and $1 \leq k \leq p - 1$. Prove that the $p \times p$ array L_k whose (i, j)th-entry is defined by

$$L_k(i, j) = ki + j \pmod p$$

defines a Latin square.

(c) Prove that when $k \neq l$ then the Latin squares L_k and L_l defined in (b) are orthogonal.

Exercise 10.2

In the lecture, the proof sketch was given for constructing a finite projective plane of order n from $n - 1$ mutually orthogonal Latin squares of order n. Show that the projective plane constructed satisfies the properties (P1) and (P2).

Exercise 10.3

Given a set system $(X, \mathcal{L})$, it is said to have *Property B* if it is possible to color each of the points of X by one of the two given colors, say red and white, such that each set $L \in \mathcal{L}$ has a point of each color. Show that the Fano plane does not have Property B.

Exercise 10.4

For natural numbers m and n, $m \leq n$, define a *Latin* $m \times n$ *rectangle* as a rectangular table with m rows and n columns with entries from the set $\{1, 2, \ldots, n\}$ such that no row or column contains the same number twice. How many Latin $2 \times n$ rectangles are there?