Combinatorics and Graph Theory I, Spring 2019, Sheet 2

Each exercise sheet gives 10 points. These points are given for:

- correctness of answers,
- conciseness of arguments, and
- readability of write-ups.

To pass the tutorial and get your credits you need a total of at least 60% of all points from exercise sheets, and 60% of all points from the two quizzes (each gives 10 points).

Please hand in your solution to this sheet at the beginning of the next tutorial on Thursday, March 7th.

Exercise 2.1
For a polynomial or power series $a(x)$, let $[x^k]a(x)$ denote the coefficient of x^k in $a(x)$. Determine the following coefficients:

(a) $[x^5](1-2x)^{-2}$

(b) $[x^4]\sqrt[3]{1+x}$

(c) $[x^3](2+x)^{3/2}/(1-x)$

(d) $[x^3](1-x+2x^2)^9$

Exercise 2.2
Find the generating functions for the following sequences and express them in closed form.

(a) $0, 0, 0, 0, -6, 6, -6, 6, -6, \ldots$

(b) $1, 0, 1, 0, 1, 0, 1, 0, \ldots$

(c) $1, 2, 1, 4, 1, 8, 1, 16, \ldots$

(d) $1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$

Exercise 2.3

(a) Let $A, B, C \subsetneq \mathbb{N}$ and let $a(x) = \sum_{i \in A} x^i$, $b(x) = \sum_{j \in B} x^j$ and $c(x) = \sum_{k \in C} x^k$ be power series. Explain why the number of solutions to the equation $i + j + k = n$ with $i \in A$, $j \in B$ and $k \in C$ is equal to the coefficient of x^n in the power series $a(x)b(x)c(x)$.

(b) Let a_n be the number of solutions to the equation

$$i + 3j + 3k = n, \qquad i \geq 0, j \geq 1, k \geq 1.$$

Find the generating function of the sequence $(a_0, a_1, a_2, \ldots)$ and derive an expression for a_n.

Exercise 2.4

(a) Find the generating function for the sequence $(a_0, a_1, a_2, \ldots)$ with $a_n = (n+1)^2$.

(b) Show that if $a(x)$ is the generating function of a sequence $(a_0, a_1, a_2, \ldots)$ then $\frac{1}{1-x}a(x)$ is the generating function of the sequence of partial sums $(a_0, a_0 + a_1, a_0 + a_1 + a2, \ldots)$.

(c) Using (a) and (b) calculate the sum $\sum_{k=1}^{n} k^2$.

Exercise 2.5

Express the nth term of the sequences given by the following recurrence relations (generalize the method used for the Fibonacci numbers done in the lecture and also in Section 12.3 of Jiří Matoušek & Jaroslav Nešetřil, Invitation to Discrete Mathematics, Second Edition):

(a) $a_0 = 2, a_1 = 3, a_{n+2} = 3a_n - 2a_{n+1}$ $(n = 0, 1, 2, \ldots)$

(b) $a_0 = 0, a_1 = 1, a_{n+2} = 4a_{n+1} - 4a_n$ $(n = 0, 1, 2, \ldots)$

(c) $a_0 = 1, a_{n+1} = 2a_n + 3$ $(n = 0, 1, 2, \ldots)$